Bayesian Statistics

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1 Normal Likelihood

1.1 Univariate Normal Distribution

Let $y_1, \dots, y_n$ be a random sample from the normal distribution $\mathcal{N}\left(\mu, \tau^{-1}\right)$.

a. Consider prior independence with prior distributions $\mu \sim \mathcal{N}\left(a, c^{-1}\right)$ and $\tau \sim \text{Gamma}(p, q)$. Calculate the conditional posterior distributions of μ and τ.

b. Consider the conjugate prior distribution $\mu \mid \tau \sim \mathcal{N}\left(a, c^{-1}\tau^{-1}\right)$, $\tau \sim \text{Gamma}(p, q)$. Calculate the conditional and marginal posterior distributions of μ and τ.

Solution.

a. The joint prior distribution may be written as follows:

$$\pi(\mu, \tau) = \pi(\mu) \cdot \pi(\tau)$$

$$
\begin{aligned}
&= \sqrt{\frac{c}{2\pi}} \exp\left\{-\frac{c(\mu-a)^2}{2}\right\} \cdot \frac{q^p}{\Gamma(p)} \tau^{p-1} e^{-q\tau} \\
&\propto \exp\left\{-c\frac{\mu^2 - 2\mu a + a^2}{2}\right\} \cdot \tau^{p-1} e^{-q\tau} \\
&\propto \exp\left\{-\frac{1}{2}c\mu^2 + ca\mu\right\} \cdot \tau^{p-1} e^{-q\tau}.
\end{aligned}
$$

The likelihood of the sample is given by:

$$
\begin{aligned}
f(y \mid \mu, \tau) &= \prod_{i=1}^{n} f(y_i \mid \mu, \tau) \\
&= \prod_{i=1}^{n} \sqrt{\frac{\tau}{2\pi}} \exp\left\{-\frac{\tau(y_i-\mu)^2}{2}\right\} \\
&\propto \tau^{\frac{n}{2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(y_i-\mu)^2\tau\right\} \\
&= \tau^{\frac{n}{2}} \exp\left\{-\sum_{i=1}^{n}\frac{y_i^2 - 2y_i\mu + \mu^2}{2}\tau\right\} \\
&= \exp\left\{-\frac{1}{2}n\tau\mu^2 + n\tau\overline{y}\mu\right\} \cdot \tau^{\frac{n}{2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} y_i^2\tau\right\}.
\end{aligned}
$$

Therefore, we get the conditional posterior distributions of μ and τ as follows:

$$
\begin{aligned}
\pi(\mu \mid \tau, y) &\propto \pi(\mu, \tau \mid y) \\
&\propto \pi(\mu, \tau) \cdot f(y \mid \mu, \tau) \\
&\propto \exp\left\{-\frac{c}{2}\mu^2 + ca\mu\right\} \cdot \exp\left\{-\frac{n\tau}{2}\mu^2 + n\tau\overline{y}\mu\right\} \\
&= \exp\left\{-\frac{1}{2}\underbrace{(c+n\tau)}_{c_n}\mu^2 + (ca + n\tau\overline{y})\,\mu\right\} \\
&= \exp\left\{-\frac{c+n\tau}{2}\mu^2 + \underbrace{(c+n\tau)}_{c_n}\underbrace{\frac{ca+n\tau\overline{y}}{c+n\tau}}_{a_n}\mu\right\},
\end{aligned}
$$

$$
\begin{aligned}
\pi(\tau \mid \mu, y) &\propto \pi(\mu, \tau) \cdot f(y \mid \mu, \tau) \\
&\propto \tau^{p-1} e^{-q\tau} \cdot \tau^{\frac{n}{2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(y_i-\mu)^2\tau\right\} \\
&= \tau^{p+\frac{n}{2}-1} \exp\left\{-\left[q + \frac{1}{2}\sum_{i=1}^{n}(y_i-\mu)^2\right]\tau\right\}.
\end{aligned}
$$

In other words,

$$
\mu \mid \tau, y \sim \mathcal{N}\left(\frac{ca+n\tau\overline{y}}{c+n\tau}, \frac{1}{c+n\tau}\right), \quad \tau \mid \mu, y \sim \text{Gamma}\left(p + \frac{n}{2}, q + \frac{1}{2}\sum_{i=1}^{n}(y_i-\mu)^2\right).
$$

We can calculate Jeffreys' prior for the univariate normal distribution as follows:

$$\log f(y \mid \mu, \tau) = \frac{1}{2}\log\tau - \frac{1}{2}\log(2\pi) - \frac{\tau(y-\mu)^2}{2},$$

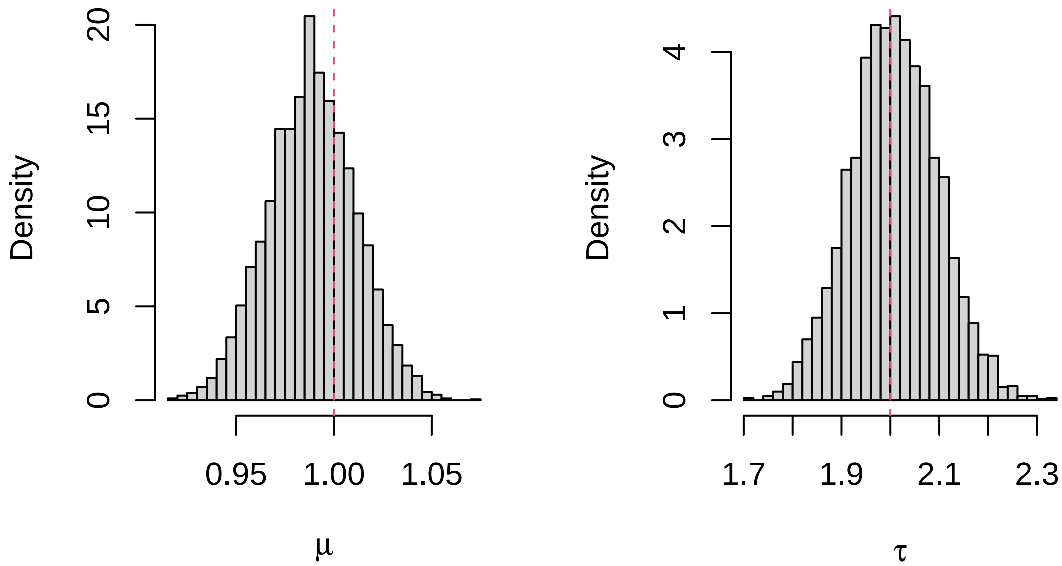
$$\frac{\partial \log f(y \mid \mu, \tau)}{\partial \mu} = \tau(y-\mu), \quad \frac{\partial \log f(y \mid \mu, \tau)}{\partial \tau} = \frac{1}{2\tau} - \frac{(y-\mu)^2}{2},$$

$$\frac{\partial^2 \log f(y \mid \mu, \tau)}{\partial \mu^2} = -\tau, \quad \frac{\partial^2 \log f(y \mid \mu, \tau)}{\partial \tau^2} = -\frac{1}{2\tau^2}, \quad \frac{\partial^2 \log f(y \mid \mu, \tau)}{\partial \mu \partial \tau} = y - \mu,$$

$$\mathcal{I}(\mu,\tau) = -\mathbb{E}\left[\frac{\partial \log f(y \mid \mu, \tau)}{\partial(\mu,\tau)\partial(\mu,\tau)}\right] = \begin{bmatrix}\tau & 0\\ 0 & \frac{1}{2\tau^2}\end{bmatrix}, \quad J(\mu,\tau) \propto \sqrt{|\mathcal{I}(\mu,\tau)|} = \sqrt{\frac{1}{2\tau}} \propto \tau^{-0.5}.$$

We observe that the improper Jeffreys' prior results for $a = c = q = 0$ and $p = 0.5$

```r
MCMCnorm = function(Y, mu0, tau0, a, c, p, q, niter, nburn) {
    n = length(Y)
    S = sum(Y)
    mu = numeric(niter)
    tau = numeric(niter)
    mu[1] = mu0
    tau[1] = tau0
    for (i in 2:niter) {
        mu[i] = rnorm(1, (c * a + tau[i - 1] * S)/(c + n * tau[i - 1]), (c +
            n * tau[i - 1])^(-0.5))
        tau[i] = rgamma(1, p + n/2, q + sum((Y - mu[i])^2)/2)
    }
    return(list(mu = mu[-(1:nburn)], tau = tau[-(1:nburn)]))
}

n = 1000
mu = 1
tau = 2
Y = rnorm(n, mu, tau^(-0.5))
posterior = MCMCnorm(Y, 0, 1, 0, 0, 0.5, 0, 5000, 1000)
par(mfrow = c(1, 2))
hist(posterior$mu, "FD", freq = FALSE, main = NA, xlab = expression(mu))
abline(v = mu, col = 2, lty = 2)
hist(posterior$tau, "FD", freq = FALSE, main = NA, xlab = expression(tau))
abline(v = tau, col = 2, lty = 2)
```



b. The joint prior distribution may be written as follows:

$$
\begin{aligned}
\pi(\mu,\tau) &= \pi(\mu \mid \tau)\cdot \pi(\tau) \\
&= \sqrt{\frac{c\tau}{2\pi}} \exp\left\{-\frac{c\tau(\mu-a)^2}{2}\right\} \cdot \frac{q^p}{\Gamma(p)}\tau^{p-1}e^{-q\tau} \\
&\propto \tau^{p+\frac{1}{2}-1} \exp\left\{-\left[q+\frac{c(\mu-a)^2}{2}\right]\tau\right\} \\
&= \tau^{\frac{1}{2}} \exp\left\{-c\tau\frac{\mu^2-2\mu a+a^2}{2}\right\} \cdot \tau^{p-1}e^{-q\tau} \\
&= \tau^{\frac{1}{2}} \exp\left\{-\frac{1}{2}c\tau\mu^2 + c\tau a\mu\right\} \cdot \tau^{p-1} \exp\left\{-\left(q+\frac{ca^2}{2}\right)\tau\right\}.
\end{aligned}
$$

Therefore, we get the joint posterior distribution of μ and τ as follows:

$$
\begin{aligned}
\pi(\mu,\tau \mid y) &\propto \pi(\mu,\tau)\cdot f(y \mid \mu,\tau) \\
&\propto \tau^{\frac{1}{2}} \exp\left\{-\frac{1}{2}c\tau\mu^2 + c\tau a\mu\right\} \cdot \exp\left\{-\frac{1}{2}n\tau\mu^2 + n\tau\overline{y}\mu\right\} \\
&\quad \times \tau^{p-1} \exp\left\{-\left(q+\frac{ca^2}{2}\right)\tau\right\} \cdot \tau^{\frac{n}{2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} y_i^2\tau\right\} \\
&= \tau^{\frac{1}{2}} \exp\left\{-\frac{1}{2}\underbrace{(c+n)}_{c_n}\tau\mu^2 + \tau\left(ca+n\overline{y}\right)\mu\right\} \cdot \tau^{p+\frac{n}{2}-1} \exp\left\{-\left(q+\frac{ca^2}{2}+\frac{1}{2}\sum_{i=1}^{n} y_i^2\right)\tau\right\} \\
&= \tau^{\frac{1}{2}} \exp\left\{-\frac{c+n}{2}\tau\mu^2 + \underbrace{(c+n)}_{c_n}\tau\underbrace{\frac{ca+n\overline{y}}{c+n}}_{a_n}\mu\right\} \cdot \tau^{p+\frac{n}{2}-1} \exp\left\{-\left(q+\frac{ca^2}{2}+\frac{1}{2}\sum_{i=1}^{n} y_i^2\right)\tau\right\} \\
&= \tau^{\frac{1}{2}} \exp\left\{-\frac{1}{2}c_n\tau\mu^2 + c_n\tau a_n\mu - \frac{1}{2}c_n\tau a_n^2 + \frac{1}{2}c_n\tau a_n^2\right\} \\
&\quad \times \tau^{p+\frac{n}{2}-1} \exp\left\{-\left(q+\frac{ca^2}{2}+\frac{1}{2}\sum_{i=1}^{n} y_i^2\right)\tau\right\}
\end{aligned}
$$

$$= \tau^{\frac{1}{2}} \exp\left\{-\frac{c_n\tau(\mu - a_n)^2}{2}\right\} \cdot \tau^{p+\frac{n}{2}-1} \exp\left\{-\left(q + \frac{ca^2}{2} + \frac{1}{2}\sum_{i=1}^n y_i^2 - \frac{c_n a_n^2}{2}\right)\tau\right\}.$$

We calculate that:

$$c_n a_n^2 = (c+n)\left(\frac{ca + n\overline{y}}{c+n}\right)^2 = \frac{(ca + n\overline{y})^2}{c+n}.$$

In other words,

$$\mu \mid \tau, y \sim \mathcal{N}\left(\frac{ca + n\overline{y}}{c+n}, \frac{\tau^{-1}}{c+n}\right), \quad \tau \mid y \sim \text{Gamma}\left(p + \frac{n}{2}, q + \frac{ca^2}{2} + \frac{1}{2}\sum_{i=1}^n y_i^2 - \frac{(ca + n\overline{y})^2}{2(c+n)}\right).$$

Furthermore, we get the conditional posterior distribution of τ as follows:

$$\begin{aligned}
\pi(\tau \mid \mu, y) &\propto \pi(\mu, \tau) \cdot f(y \mid \mu, \tau) \\
&\propto \tau^{p+\frac{1}{2}-1} \exp\left\{-\left[q + \frac{c(\mu - a)^2}{2}\right]\tau\right\} \cdot \tau^{\frac{n}{2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^n (y_i - \mu)^2 \tau\right\} \\
&= \tau^{p+\frac{n+1}{2}-1} \exp\left\{-\left[q + \frac{c(\mu - a)^2}{2} + \frac{1}{2}\sum_{i=1}^n (y_i - \mu)^2\right]\tau\right\}.
\end{aligned}$$

In other words,

$$\tau \mid \mu, y \sim \text{Gamma}\left(p + \frac{n+1}{2}, q + \frac{c(\mu - a)^2}{2} + \frac{1}{2}\sum_{i=1}^n (y_i - \mu)^2\right).$$

Definition 1.1. We say that a random variable X follows the generalized Student's t distribution with mean $\mu \in \mathbb{R}$, variance $\sigma^2 > 0$ and $\nu > 0$ degrees of freedom, i.e. $X \sim t_\nu\left(\mu, \sigma^2\right)$, if it has the following probability density function:

$$f_X(x \mid \mu, \sigma^2, \nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\nu\pi\sigma^2}} \left[1 + \frac{1}{\nu\sigma^2}(x - \mu)^2\right]^{-\frac{\nu+1}{2}}, \quad x \in \mathbb{R}.$$

Finally, we define:

$$p_n = p + \frac{n}{2}, \quad q_n = q + \frac{ca^2}{2} + \frac{1}{2}\sum_{i=1}^n y_i^2 - \frac{c_n a_n^2}{2}.$$

Then, we calculate the marginal posterior distribution of μ as follows:

$$\begin{aligned}
\pi(\mu \mid y) &= \int \pi(\mu, \tau \mid y) d\tau \\
&\propto \int \tau^{p+\frac{n+1}{2}-1} \exp\left\{-\left[q + \frac{c(\mu - a)^2}{2} + \frac{1}{2}\sum_{i=1}^n (y_i - \mu)^2\right]\tau\right\} d\tau \\
&\propto \left[q + \frac{c(\mu - a)^2}{2} + \frac{1}{2}\sum_{i=1}^n (y_i - \mu)^2\right]^{-\left(p+\frac{n+1}{2}\right)} \\
&= \left[q + \frac{ca^2}{2} + \frac{1}{2}\sum_{i=1}^n y_i^2 + \frac{c+n}{2}\mu^2 - (ca + n\overline{y})\mu\right]^{-p_n-\frac{1}{2}} \\
&= \left[q + \frac{ca^2}{2} + \frac{1}{2}\sum_{i=1}^n y_i^2 - \frac{c_n a_n^2}{2} + \frac{c_n(\mu - a_n)^2}{2}\right]^{-\frac{2p_n+1}{2}}
\end{aligned}$$

$$\propto \left[1 + \frac{c_n (\mu - a_n)^2}{2q_n}\right]^{-\frac{2p_n+1}{2}}$$

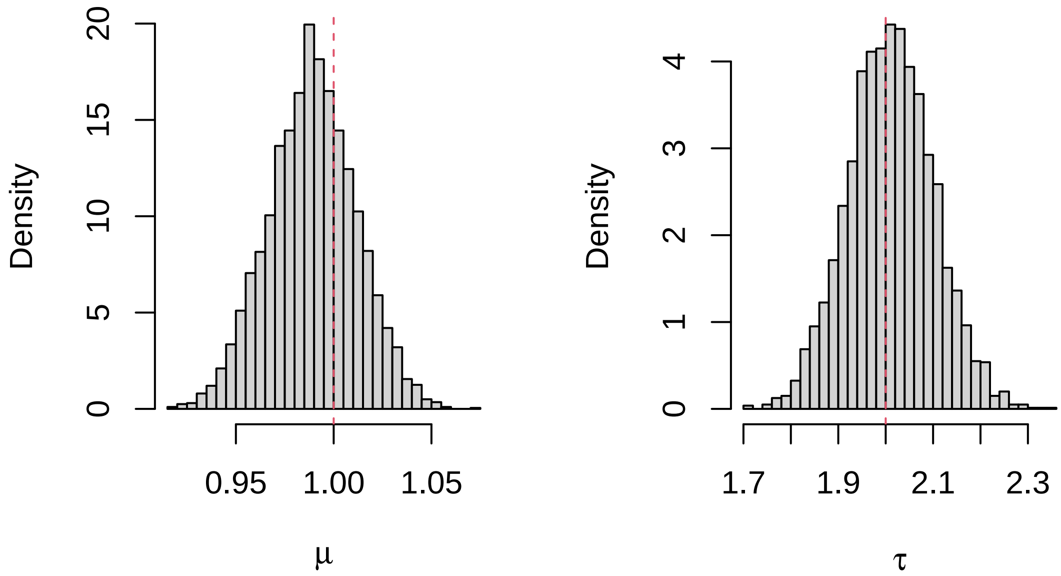
$$= \left[1 + \frac{1}{2p_n} \frac{(\mu - a_n)^2}{\frac{q_n}{c_n p_n}}\right]^{-\frac{2p_n+1}{2}}.$$

In other words,

$$\mu \mid y \sim t_{2p_n}\left(a_n, \frac{q_n}{c_n p_n}\right).$$

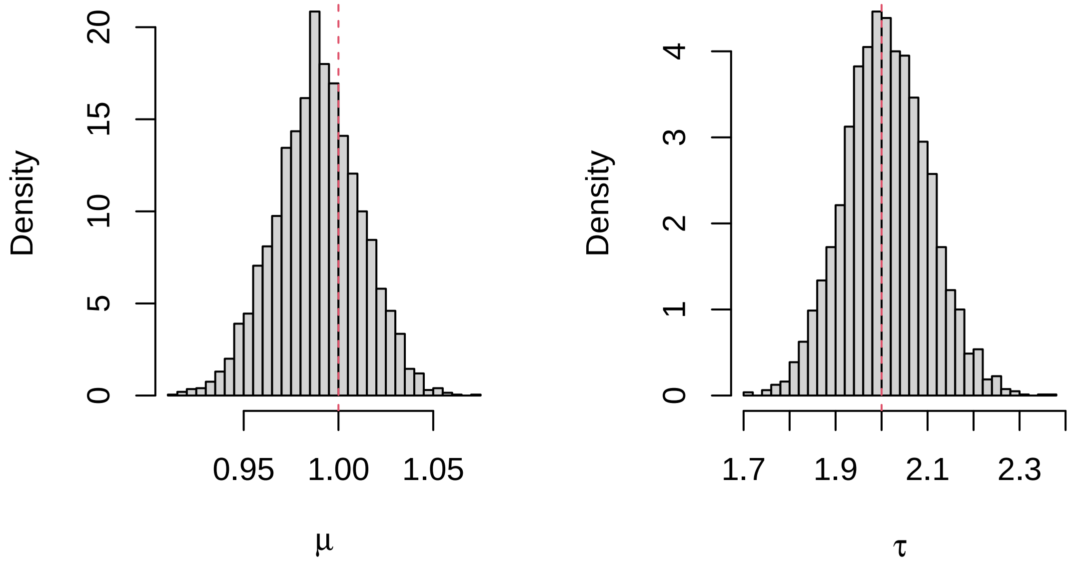
First, we implement a Gibbs sampler which alternately simulates from the conditional posterior distributions of μ and τ.

```
MCMCnorm = function(Y, mu0, tau0, a, c, p, q, niter, nburn) {
    n = length(Y)
    S = sum(Y)
    cn = c + n
    an = (c * a + S)/cn
    mu = numeric(niter)
    tau = numeric(niter)
    mu[1] = mu0
    tau[1] = tau0
    for (i in 2:niter) {
        mu[i] = rnorm(1, an, (cn * tau[i - 1])^(-0.5))
        tau[i] = rgamma(1, p + (n + 1)/2, q + c * (mu[i] - a)^2/2 + sum((Y -
            mu[i])^2)/2)
    }
    return(list(mu = mu[-(1:nburn)], tau = tau[-(1:nburn)]))
}

posterior = MCMCnorm(Y, 0, 1, 0, 0, 0.5, 0, 5000, 1000)
par(mfrow = c(1, 2))
hist(posterior$mu, "FD", freq = FALSE, main = NA, xlab = expression(mu))
abline(v = mu, col = 2, lty = 2)
hist(posterior$tau, "FD", freq = FALSE, main = NA, xlab = expression(tau))
abline(v = tau, col = 2, lty = 2)
```



Next, we implement the composition method which first simulates from the marginal posterior distribution of τ and then from the conditional posterior distribution of μ.

```
CMnorm = function(Y, a, c, p, q, niter) {
    n = length(Y)
    S = sum(Y)
    cn = c + n
    an = (c * a + S)/cn
    pn = p + n/2
    qn = q + c * a^2/2 + sum(Y^2)/2 - cn * an^2/2
    mu = numeric(niter)
    tau = numeric(niter)
    for (i in 1:niter) {
        tau[i] = rgamma(1, pn, qn)
        mu[i] = rnorm(1, an, (cn * tau[i])^(-0.5))
    }
    return(list(mu = mu, tau = tau))
}

posterior = CMnorm(Y, 0, 0, 0.5, 0, 4000)
par(mfrow = c(1, 2))
hist(posterior$mu, "FD", freq = FALSE, main = NA, xlab = expression(mu))
abline(v = mu, col = 2, lty = 2)
hist(posterior$tau, "FD", freq = FALSE, main = NA, xlab = expression(tau))
abline(v = tau, col = 2, lty = 2)
```



1.2 Generalized Student's t Distribution

Let $y_1, \dots, y_n$ be a random sample from the generalized Student's t distribution with mean $\mu \in \mathbb{R}$, precision $\tau > 0$ and $\nu > 0$ degrees of freedom, that is:

$$f(y_i \mid \mu, \tau, \nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \sqrt{\frac{\tau}{\nu\pi}} \left[1 + \frac{1}{\nu}\tau(y_i - \mu)^2\right]^{-\frac{\nu+1}{2}}, \quad y_i \in \mathbb{R}.$$

Consider the random variables $W_i \sim \mathcal{N}\left(0, \tau^{-1}\right)$ and $V_i \sim \chi^2_\nu \equiv \text{Gamma}\left(\frac{\nu}{2}, \frac{1}{2}\right)$. Then, we observe that:

$$Y_i \overset{d}{=} \frac{W_i}{\sqrt{\frac{V_i}{\nu}}} + \mu.$$

We let $Z_i = \frac{V_i}{\nu}$. Then, $Z_i \sim \text{Gamma}\left(\frac{\nu}{2}, \frac{\nu}{2}\right)$. We observe that:

$$Y_i \mid z_i \overset{d}{=} \frac{W_i}{\sqrt{z_i}} + \mu \sim \mathcal{N}\left(\mu, \tau^{-1} z_i^{-1}\right).$$

Suppose that the degrees of freedom ν are known and that the parameters μ, τ are a priori independent with prior distributions $\mu \sim \mathcal{N}\left(a, c^{-1}\right)$, $\tau \sim \text{Gamma}(p, q)$. Calculate the conditional posterior distributions of the parameters μ, τ and the latent variables z_i.

Solution.

The joint prior distribution may be written as follows:

$$\begin{aligned}
\pi(\mu, \tau) &= \pi(\mu) \cdot \pi(\tau) \\
&= \sqrt{\frac{c}{2\pi}} \exp\left\{-\frac{c(\mu - a)^2}{2}\right\} \cdot \frac{q^p}{\Gamma(p)} \tau^{p-1} e^{-q\tau} \\
&\propto \exp\left\{-c\frac{\mu^2 - 2\mu a + a^2}{2}\right\} \cdot \tau^{p-1} e^{-q\tau}
\end{aligned}$$

$$\propto \exp\left\{-\frac{1}{2}c\mu^2 + ca\mu\right\} \cdot \tau^{p-1}e^{-q\tau}.$$

We define:

$$\overline{zy} = \frac{1}{n}\sum_{i=1}^{n} z_i y_i.$$

The complete-data likelihood, i.e. the joint likelihood of the observed variables y_i and the latent variables z_i, is given by:

$$\begin{aligned}
f(y, z \mid \mu, \tau) &= \prod_{i=1}^{n} f(y_i, z_i \mid \mu, \tau) \\
&= \prod_{i=1}^{n} f(z_i) f(y_i \mid z_i, \mu, \tau) \\
&= \prod_{i=1}^{n} \frac{\left(\frac{\nu}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} z_i^{\frac{\nu}{2}-1} e^{-\frac{\nu}{2}z_i} \sqrt{\frac{\tau z_i}{2\pi}} \exp\left\{-\frac{\tau z_i (y_i-\mu)^2}{2}\right\} \\
&\propto \tau^{\frac{n}{2}} \cdot \prod_{i=1}^{n} z_i^{\frac{\nu+1}{2}-1} \exp\left\{-\sum_{i=1}^{n} \frac{\nu + \tau(y_i-\mu)^2}{2} z_i\right\} \\
&= \tau^{\frac{n}{2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} z_i (y_i-\mu)^2 \tau\right\} \cdot \prod_{i=1}^{n} z_i^{\frac{\nu+1}{2}-1} \exp\left\{-\frac{\nu}{2}\sum_{i=1}^{n} z_i\right\} \\
&= \tau^{\frac{n}{2}} \exp\left\{-\sum_{i=1}^{n} \frac{y_i^2 - 2y_i\mu + \mu^2}{2} z_i \tau\right\} \cdot \prod_{i=1}^{n} z_i^{\frac{\nu+1}{2}-1} \exp\left\{-\frac{\nu}{2}\sum_{i=1}^{n} z_i\right\} \\
&= \exp\left\{-\frac{1}{2}n\tau\overline{z}\mu^2 + n\tau\overline{zy}\mu\right\} \cdot \tau^{\frac{n}{2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} z_i y_i^2 \tau\right\} \cdot \prod_{i=1}^{n} z_i^{\frac{\nu+1}{2}-1} \exp\left\{-\frac{\nu}{2}\sum_{i=1}^{n} z_i\right\}.
\end{aligned}$$

Therefore, we get the conditional posterior distributions of μ and τ as follows:

$$\begin{aligned}
\pi(\mu \mid \tau, z, y) &\propto \pi(\mu, \tau, z \mid y) \\
&\propto \pi(\mu, \tau) \cdot f(y, z \mid \mu, \tau) \\
&\propto \exp\left\{-\frac{1}{2}c\mu^2 + ca\mu\right\} \cdot \exp\left\{-\frac{1}{2}n\tau\overline{z}\mu^2 + n\tau\overline{zy}\mu\right\} \\
&= \exp\left\{-\frac{1}{2}\underbrace{(c + n\tau\overline{z})}_{c_n}\mu^2 + (ca + n\tau\overline{zy})\,\mu\right\} \\
&= \exp\left\{-\frac{c + n\tau\overline{z}}{2}\mu^2 + \underbrace{(c + n\tau\overline{z})}_{c_n}\underbrace{\frac{ca + n\tau\overline{zy}}{c + n\tau\overline{z}}}_{a_n}\mu\right\},
\end{aligned}$$

$$\begin{aligned}
\pi(\tau \mid \mu, z, y) &\propto \pi(\mu, \tau) \cdot f(y, z \mid \mu, \tau) \\
&\propto \tau^{p-1}e^{-q\tau} \cdot \tau^{\frac{n}{2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} z_i (y_i-\mu)^2 \tau\right\}
\end{aligned}$$

$$= \tau^{p+\frac{n}{2}-1} \exp\left\{ - \left[q + \frac{1}{2} \sum_{i=1}^{n} z_i (y_i - \mu)^2 \right] \tau \right\}.$$

Furthermore, we get the conditional posterior distribution of the latent variables z_i as follows:

$$f(z_i \mid y_i, \mu, \tau) \propto f(y_i, z_i \mid \mu, \tau) \propto z_i^{\frac{\nu+1}{2}-1} \exp\left\{ -\frac{\nu + \tau(y_i - \mu)^2}{2} z_i \right\}.$$

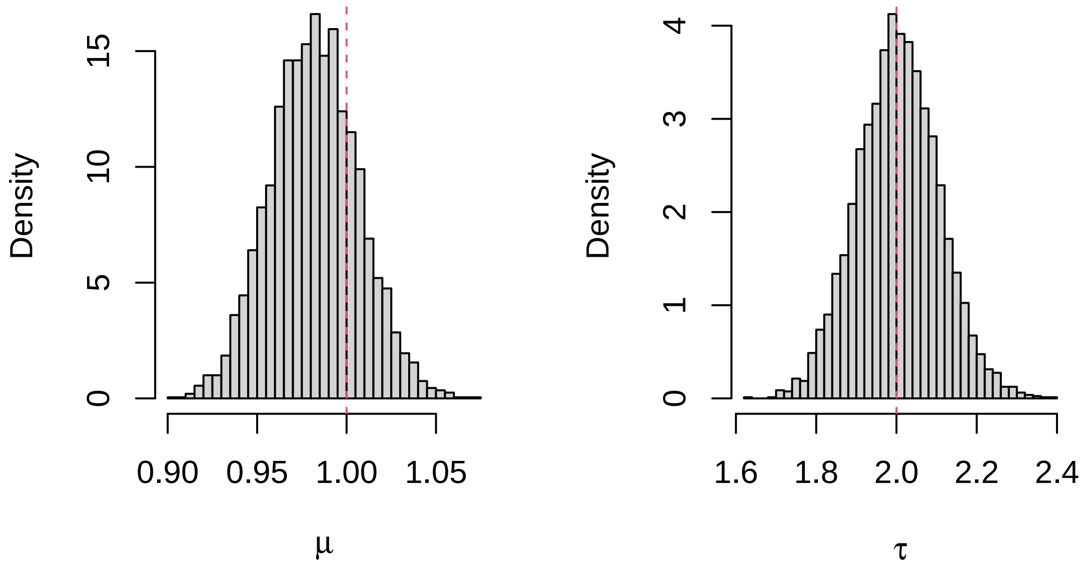
In other words,

$$\mu \mid \tau, z, y \sim \mathcal{N}\left(\frac{ca + n\tau\overline{zy}}{c + n\tau\overline{z}}, \frac{1}{c + n\tau\overline{z}} \right), \quad \tau \mid \mu, z, y \sim \text{Gamma}\left(p + \frac{n}{2}, q + \frac{1}{2} \sum_{i=1}^{n} z_i (y_i - \mu)^2 \right),$$

$$z_i \mid y_i, \mu, \tau \sim \text{Gamma}\left(\frac{\nu + 1}{2}, \frac{\nu + \tau(y_i - \mu)^2}{2} \right).$$

```
MCMCt = function(Y, mu0, tau0, nu, a, c, p, q, niter, nburn) {
    n = length(Y)
    mu = numeric(niter)
    tau = numeric(niter)
    Z = matrix(0, niter, n)
    mu[1] = mu0
    tau[1] = tau0
    Z[1, ] = rgamma(n, (nu + 1)/2, (nu + tau[1] * (Y - mu[1])^2)/2)
    for (i in 2:niter) {
        mu[i] = rnorm(1, (c * a + tau[i - 1] * sum(Z[i - 1, ] * Y))/(c + tau[i -
            1] * sum(Z[i - 1, ])), (c + n * tau[i - 1])^(-0.5))
        tau[i] = rgamma(1, p + n/2, q + sum(Z[i - 1, ] * (Y - mu[i])^2)/2)
        Z[i, ] = rgamma(n, (nu + 1)/2, (nu + tau[i] * (Y - mu[i])^2)/2)
    }
    return(list(mu = mu[-(1:nburn)], tau = tau[-(1:nburn)], Z = Z[-(1:nburn),
        ]))
}

library(mvtnorm)
n = 1000
mu = 1
tau = 2
nu = 10
Y = rmvt(n, matrix(tau^(-1)), nu, mu)
posterior = MCMCt(Y, 0, 1, nu, 0, 0, 0.5, 0, 5000, 1000)
par(mfrow = c(1, 2))
hist(posterior$mu, "FD", freq = FALSE, main = NA, xlab = expression(mu))
abline(v = mu, col = 2, lty = 2)
hist(posterior$tau, "FD", freq = FALSE, main = NA, xlab = expression(tau))
```

```r
abline(v = tau, col = 2, lty = 2)
```



Alternatively, we can implement a Random Walk Metropolis-Hastings algorithm. We consider the proposed random variable $\mu^* \mid \mu_{\ell-1} \sim \mathcal{N}\left(\mu_{\ell-1}, \sigma_\mu^2\right)$ for the parameter $\mu \in \mathbb{R}$ with the following acceptance probability:

$$A\left(\mu_{\ell-1}, \mu^*\right) = \min\left\{\frac{\pi\left(\mu^*\right)}{\pi\left(\mu_{\ell-1}\right)} \frac{f\left(y \mid \mu^*, \tau_{\ell-1}\right)}{f\left(y \mid \mu_{\ell-1}, \tau_{\ell-1}\right)}, 1\right\}.$$

We consider the proposed random variable $\tau^* \mid \tau_{\ell-1} \sim \text{Lognormal}\left(\log \tau_{\ell-1}, \sigma_\tau^2\right)$ for the parameter $\tau > 0$ with the following conditional probability density function:

$$f\left(\tau^\star \mid \tau_{\ell-1}\right) = \frac{1}{\sqrt{2\pi\sigma_\tau^2 \tau_{\ell-1}^2}} \exp\left\{-\frac{\left(\log \tau^* - \log \tau_{\ell-1}\right)^2}{2\sigma_\tau^2}\right\}.$$

Then, the acceptance probability of the value τ^* is given by:

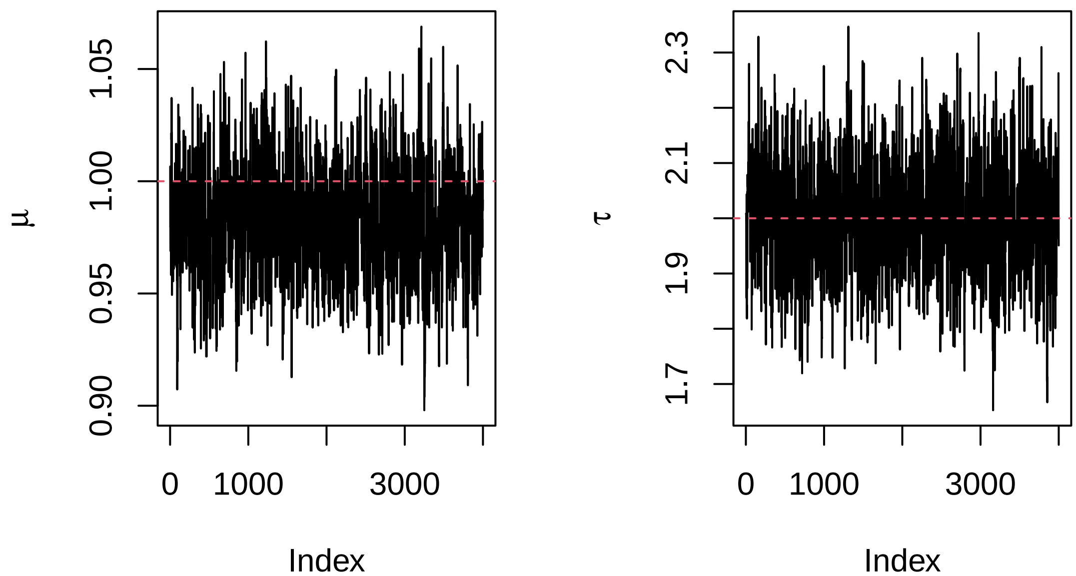
$$A\left(\tau_{\ell-1}, \tau^*\right) = \min\left\{\frac{\pi\left(\tau^*\right)}{\pi\left(\tau_{\ell-1}\right)} \frac{f\left(y \mid \mu_\ell, \tau^*\right)}{f\left(y \mid \mu_\ell, \tau_{\ell-1}\right)} \frac{\tau^*}{\tau_{\ell-1}}, 1\right\}.$$

If $Z \sim \mathcal{N}\left(0, \sigma_\tau^2\right)$, then we know that $\tau^* \mid \tau_{\ell-1} \sim \tau_{\ell-1} e^Z$. Therefore, this proposal is called a multiplicative random walk and may be equivalently written as a random walk on the log scale in the following manner:

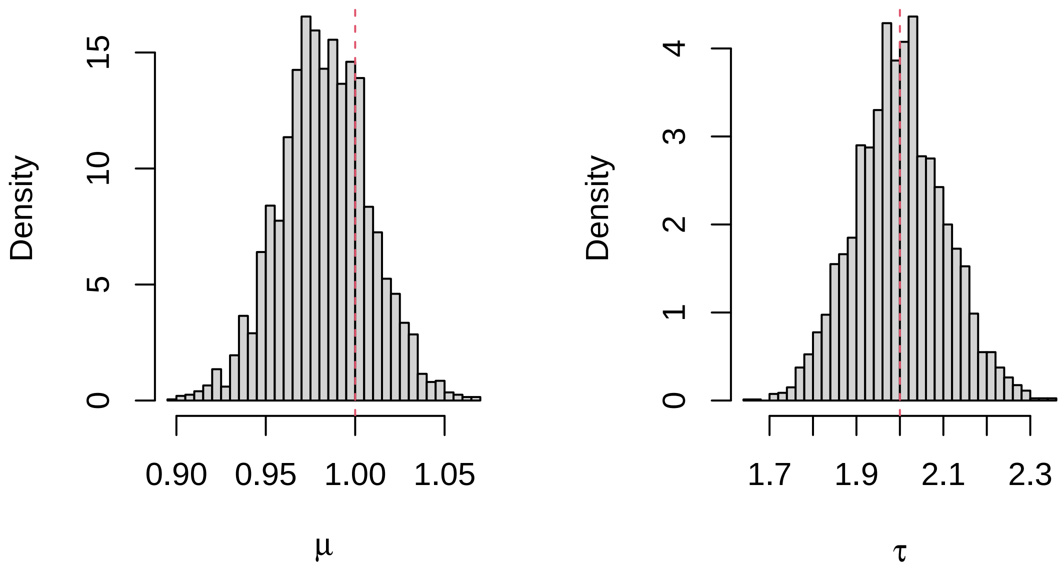
$$\log \tau^* \mid \tau_{\ell-1} \sim \mathcal{N}\left(\log \tau_{\ell-1}, \sigma_\tau^2\right).$$

```r
RWMHt = function(Y, mu0, tau0, nu, musd, tausd, niter, nburn) {
    library(mvtnorm)
    mu = numeric(niter)
    tau = numeric(niter)
    mu[1] = mu0
    tau[1] = tau0
    for (i in 2:niter) {
```

```
        mustar = rnorm(1, mu[i - 1], musd)
        logA = sum(dmvt(Y, mustar, matrix(tau[i - 1]^(-1)), nu) - dmvt(Y, mu[i -
            1], matrix(tau[i - 1]^(-1)), nu))
        mu[i] = ifelse(log(runif(1)) < logA, mustar, mu[i - 1])
        taustar = tau[i - 1] * exp(rnorm(1, sd = tausd))
        logA = log(taustar/tau[i - 1])/2 + sum(dmvt(Y, mu[i], matrix(taustar^(-1)),
            nu) - dmvt(Y, mu[i], matrix(tau[i - 1]^(-1)), nu))
        tau[i] = ifelse(log(runif(1)) < logA, taustar, tau[i - 1])
    }
    return(list(mu = mu[-(1:nburn)], tau = tau[-(1:nburn)]))
}

posterior = RWMHt(Y, 0, 1, nu, 0.05, 0.1, 5000, 1000)
par(mfrow = c(1, 2))
plot(posterior$mu, type = "l", ylab = expression(mu))
abline(h = mu, col = 2, lty = 2)
plot(posterior$tau, type = "l", ylab = expression(tau))
abline(h = tau, col = 2, lty = 2)
```




```
hist(posterior$mu, "FD", freq = FALSE, main = NA, xlab = expression(mu))
abline(v = mu, col = 2, lty = 2)
hist(posterior$tau, "FD", freq = FALSE, main = NA, xlab = expression(tau))
abline(v = tau, col = 2, lty = 2)
```



1.3 One-Way Analysis of Variance Model

Consider the analysis of variance model $y_{ij} = \mu_i + \varepsilon_{ij}$, where $\varepsilon_i \sim \mathcal{N}\left(0, \tau^{-1}\right)$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n_i$. We consider prior independence with prior distributions $\mu_i \sim \mathcal{N}\left(a, c^{-1}\right)$ and $\tau \sim \text{Gamma}(p, q)$. Calculate the conditional posterior distributions of μ_i and τ.

Solution.

The joint prior distribution may be written as follows:

$$\begin{aligned}
\pi(\mu, \tau) &= \pi(\tau) \cdot \prod_{i=1}^{m} \pi(\mu_i) \\
&= \frac{q^p}{\Gamma(p)} \tau^{p-1} e^{-q\tau} \cdot \prod_{i=1}^{m} \sqrt{\frac{c}{2\pi}} \exp\left\{-\frac{c(\mu_i - a)^2}{2}\right\} \\
&\propto \exp\left\{-c \sum_{i=1}^{m} \frac{\mu_i^2 - 2a\mu_i + a^2}{2}\right\} \cdot \tau^{p-1} e^{-q\tau} \\
&\propto \exp\left\{-\frac{c}{2} \sum_{i=1}^{m} \mu_i^2 + ca \sum_{i=1}^{m} \mu_i\right\} \cdot \tau^{p-1} e^{-q\tau}.
\end{aligned}$$

We define:

$$n = \sum_{i=1}^{m} N_i, \quad \overline{y}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij}.$$

Then, the likelihood of the sample is given by:

$$\begin{aligned}
f(y \mid \mu, \tau) &= \prod_{i=1}^{m} \prod_{j=1}^{n_i} f(y_{ij} \mid \mu, \tau) \\
&= \prod_{i=1}^{m} \prod_{j=1}^{n_i} \sqrt{\frac{\tau}{2\pi}} \exp\left\{-\frac{\tau(y_{ij} - \mu_i)^2}{2}\right\}
\end{aligned}$$

$$
\begin{aligned}
&\propto \tau^{\frac{n}{2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n_i}(y_{ij}-\mu_i)^2\tau\right\} \\
&= \tau^{\frac{n}{2}} \exp\left\{-\sum_{i=1}^{m}\sum_{j=1}^{n_i}\frac{y_{ij}^2-2y_{ij}\mu_i+\mu_i^2}{2}\tau\right\} \\
&= \exp\left\{-\frac{1}{2}\tau\sum_{i=1}^{m}N_i\mu_i^2+\tau\sum_{i=1}^{m}N_i\overline{y}_i\mu_i\right\}\cdot\tau^{\frac{n}{2}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n_i}y_{ij}^2\tau\right\}.
\end{aligned}
$$

Therefore, we get the conditional posterior distributions of μ_i and τ as follows:

$$
\begin{aligned}
\pi(\mu_i \mid \tau, y) &\propto \pi(\mu, \tau \mid y) \\
&\propto \pi(\mu,\tau)\cdot f(y \mid \mu,\tau) \\
&\propto \exp\left\{-\frac{c}{2}\mu_i^2+ca\mu_i\right\}\cdot\exp\left\{-\frac{1}{2}n_i\tau\mu_i^2+n_i\tau\overline{y}_i\mu_i\right\} \\
&= \exp\left\{-\frac{1}{2}\underbrace{(c+n_i\tau)}_{c_n}\mu_i^2+(ca+n_i\tau\overline{y}_i)\,\mu_i\right\} \\
&= \exp\left\{-\frac{c+n_i\tau}{2}\mu_i^2+\underbrace{(c+n_i\tau)}_{c_n}\underbrace{\frac{ca+n_i\tau\overline{y}_i}{c+n_i\tau}}_{a_n}\mu_i\right\},
\end{aligned}
$$

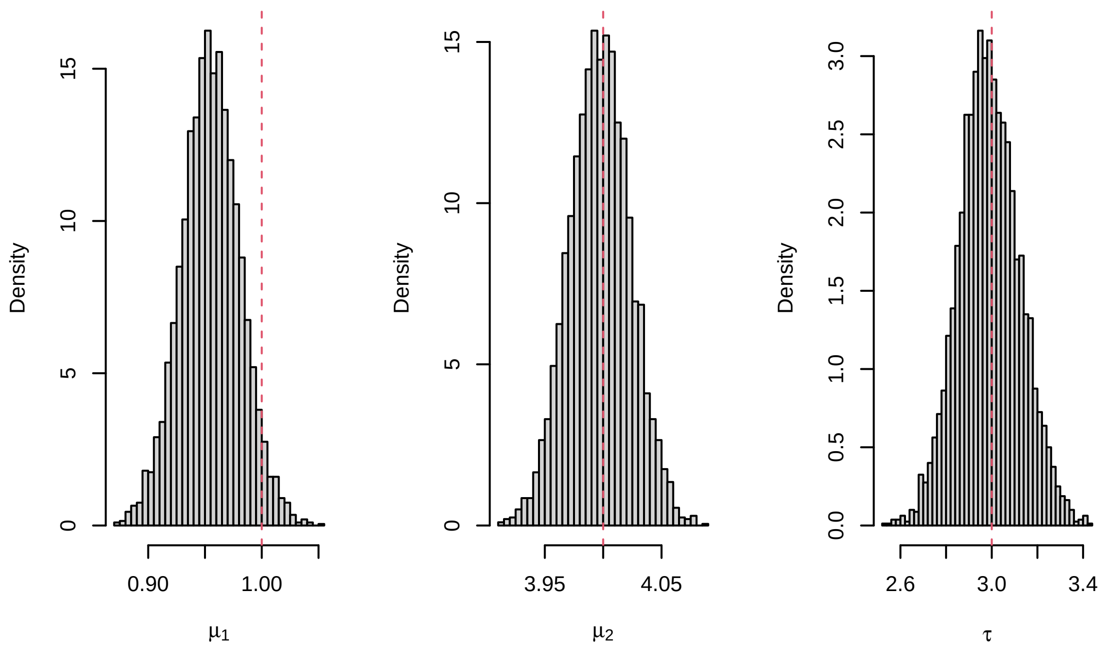
$$
\begin{aligned}
\pi(\tau \mid \mu, y) &\propto \pi(\mu,\tau)\cdot f(y \mid \mu,\tau) \\
&\propto \tau^{p-1}e^{-q\tau}\cdot\tau^{\frac{n}{2}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n_i}(y_{ij}-\mu_i)^2\tau\right\} \\
&= \tau^{p+\frac{n}{2}-1}\exp\left\{-\left[q+\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n_i}(y_{ij}-\mu_i)^2\right]\tau\right\}.
\end{aligned}
$$

In other words,

$$
\mu_i \mid \tau, y \sim \mathcal{N}\left(\frac{ca+n_i\tau\overline{y}_i}{c+n_i\tau},\frac{1}{c+n_i\tau}\right), \quad \tau \mid \mu, y \sim \text{Gamma}\left(p+\frac{n}{2}, q+\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n_i}(y_{ij}-\mu_i)^2\right).
$$

```r
MCMCanova = function(Y, X, mu0, tau0, a, c, p, q, niter, nburn) {
    n = length(Y)
    m = length(levels(X))
    N = table(X)
    S = aggregate(Y ~ X, FUN = sum)[, 2]
    mu = matrix(0, niter, m)
    tau = numeric(niter)
    mu[1, ] = mu0
    tau[1] = tau0
    for (i in 2:niter) {
```

```
        mu[i, ] = rnorm(m, (c * a + tau[i - 1] * S)/(c + N * tau[i - 1]), (c +
            N * tau[i - 1])^(-0.5))
        tau[i] = rgamma(1, p + n/2, q + sum((Y - mu[i, X])^2)/2)
    }
    return(list(tau = tau[-(1:nburn)], mu = mu[-(1:nburn), ]))
}

n = 1000
m = 2
tau = 3
mu = c(1, 4)
X = factor(sample(m, n, replace = TRUE), levels = 1:m)
Y = rnorm(n, mu[X], tau^(-0.5))
posterior = MCMCanova(Y, X, numeric(m), 1, 0, 0, 0.5, 0, 5000, 1000)
par(mfrow = c(1, 3))
hist(posterior$mu[, 1], "FD", freq = FALSE, main = NA, xlab = expression(mu[1]))
abline(v = mu[1], col = 2, lty = 2)
hist(posterior$mu[, 2], "FD", freq = FALSE, main = NA, xlab = expression(mu[2]))
abline(v = mu[2], col = 2, lty = 2)
hist(posterior$tau, "FD", freq = FALSE, main = NA, xlab = expression(tau))
abline(v = tau, col = 2, lty = 2)
```



1.4 Linear Model with Student's t Error Term

Consider the linear model $y_i = x_i^{\mathrm{T}}\beta + \varepsilon_i$, where $\beta \in \mathbb{R}^k$ and the error term ε_i follows the generalized Student's t distribution with mean 0, precision $\tau > 0$ and $\nu > 0$ degrees of freedom, that is:

$$f(y_i \mid \beta, \tau, \nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\sqrt{\frac{\tau}{\nu\pi}}\left[1 + \frac{1}{\nu}\tau\left(y_i - x_i^{\mathrm{T}}\beta\right)^2\right]^{-\frac{\nu+1}{2}}, \quad y_i \in \mathbb{R}.$$

Consider the random variables $W_i \sim \mathcal{N}\left(0, \tau^{-1}\right)$ and $V_i \sim \chi^2_\nu \equiv \text{Gamma}\left(\frac{\nu}{2}, \frac{1}{2}\right)$. Then, we observe that:

$$Y_i \overset{d}{=} \frac{W_i}{\sqrt{\frac{V_i}{\nu}}} + x_i^{\mathrm{T}}\beta.$$

We let $Z_i = \frac{V_i}{\nu}$. Then, $Z_i \sim \text{Gamma}\left(\frac{\nu}{2}, \frac{\nu}{2}\right)$. We observe that:

$$Y_i \mid z_i \overset{d}{=} \frac{W_i}{\sqrt{z_i}} + x_i^{\mathrm{T}}\beta \sim \mathcal{N}\left(x_i^{\mathrm{T}}\beta, \tau^{-1}z_i^{-1}\right).$$

Suppose that the degrees of freedom ν are known and consider the conditionally conjugate prior distribution $\beta \mid \tau \sim \mathcal{N}_k\left(a, \tau^{-1}C^{-1}\right)$, $\tau \sim \text{Gamma}(p, q)$. Calculate the conditional posterior distributions $\pi(\beta, \tau \mid z, y)$, $\pi(\tau \mid \beta, z, y)$ and $f(z_i \mid y_i, \beta, \tau)$.

Solution.

The joint prior distribution may be written as follows:

$$\begin{aligned}
\pi(\beta, \tau) &= \pi(\beta \mid \tau) \cdot \pi(\tau) \\
&= (2\pi)^{-\frac{k}{2}}\left|\tau^{-1}C^{-1}\right|^{-\frac{1}{2}} \exp\left\{-\frac{\tau(\beta - a)^{\mathrm{T}}C(\beta - a)}{2}\right\} \cdot \frac{q^p}{\Gamma(p)}\tau^{p-1}e^{-q\tau} \\
&\propto \tau^{p+\frac{k}{2}-1} \exp\left\{-\left[q + \frac{(\beta - a)^{\mathrm{T}}C(\beta - a)}{2}\right]\tau\right\} \\
&= \tau^{\frac{k}{2}} \exp\left\{-\tau\frac{\beta^{\mathrm{T}}C\beta - 2\beta^{\mathrm{T}}Ca + a^{\mathrm{T}}Ca}{2}\right\} \cdot \tau^{p-1}e^{-q\tau} \\
&= \tau^{\frac{k}{2}} \exp\left\{-\frac{\beta^{\mathrm{T}}\tau C\beta}{2} + \beta^{\mathrm{T}}\tau Ca\right\} \cdot \tau^{p-1} \exp\left\{-\left(q + \frac{a^{\mathrm{T}}Ca}{2}\right)\tau\right\}.
\end{aligned}$$

The complete-data likelihood, i.e. the joint likelihood of the observed variables y_i and the latent variables z_i, is given by:

$$\begin{aligned}
f(y, z \mid \beta, \tau) &= \prod_{i=1}^{n} f(y_i, z_i \mid \beta, \tau) \\
&= \prod_{i=1}^{n} f(z_i)f(y_i \mid z_i, \beta, \tau) \\
&= \prod_{i=1}^{n} \frac{\left(\frac{\nu}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} z_i^{\frac{\nu}{2}-1}e^{-\frac{\nu}{2}z_i}\sqrt{\frac{\tau z_i}{2\pi}} \exp\left\{-\frac{\tau z_i\left(y_i - x_i^{\mathrm{T}}\beta\right)^2}{2}\right\} \\
&\propto \tau^{\frac{n}{2}} \cdot \prod_{i=1}^{n} z_i^{\frac{\nu+1}{2}-1} \exp\left\{-\sum_{i=1}^{n}\frac{\nu + \tau\left(y_i - x_i^{\mathrm{T}}\beta\right)^2}{2}z_i\right\}.
\end{aligned}$$

We define $y = (y_1, \ldots, y_n)^{\mathrm{T}} \in \mathbb{R}^n$, the design matrix $X = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^{n\times k}$ and the diagonal weight matrix $Z = \mathrm{diag}(z_1, \ldots, z_n) \in \mathbb{R}^{n\times n}$. Then, we observe that $y \mid z \sim N_n\left(X\beta, \tau^{-1}Z^{-1}\right)$. In other words, the complete-data likelihood is given by:

$$
\begin{aligned}
f(y, z \mid \beta, \tau) &= f(z) \cdot f(y \mid z, \beta, \tau) \\
&= f(y \mid z, \beta, \tau) \cdot \prod_{i=1}^{n} f(z_i) \\
&= (2\pi)^{-\frac{n}{2}} \left|\tau^{-1}Z^{-1}\right|^{-\frac{1}{2}} \exp\left\{-\frac{\tau(y - X\beta)^{\mathrm{T}}Z(y - X\beta)}{2}\right\} \cdot \prod_{i=1}^{n} \frac{\left(\frac{\nu}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} z_i^{\frac{\nu}{2}-1} e^{-\frac{\nu}{2}z_i} \\
&\propto \tau^{\frac{n}{2}} \exp\left\{-\frac{(y - X\beta)^{\mathrm{T}}Z(y - X\beta)}{2}\tau\right\} \cdot |Z|^{\frac{1}{2}} \prod_{i=1}^{n} z_i^{\frac{\nu}{2}-1} \exp\left\{-\frac{\nu}{2}\sum_{i=1}^{n} z_i\right\} \\
&= \tau^{\frac{n}{2}} \exp\left\{-\frac{y^{\mathrm{T}}Zy - 2\beta^{\mathrm{T}}X^{\mathrm{T}}Zy + \beta^{\mathrm{T}}X^{\mathrm{T}}ZX\beta}{2}\tau\right\} \cdot \prod_{i=1}^{n} z_i^{\frac{1}{2}} \prod_{i=1}^{n} z_i^{\frac{\nu}{2}-1} \exp\left\{-\frac{\nu}{2}\sum_{i=1}^{n} z_i\right\} \\
&= \exp\left\{-\frac{\beta^{\mathrm{T}}\tau X^{\mathrm{T}}ZX\beta}{2} + \beta^{\mathrm{T}}\tau X^{\mathrm{T}}Zy\right\} \cdot \tau^{\frac{n}{2}} \exp\left\{-\frac{y^{\mathrm{T}}Zy}{2}\tau\right\} \\
&\quad \times \prod_{i=1}^{n} z_i^{\frac{\nu+1}{2}-1} \exp\left\{-\frac{\nu}{2}\sum_{i=1}^{n} z_i\right\}.
\end{aligned}
$$

Therefore, we get the joint conditional posterior distribution of β and τ as follows:

$$
\begin{aligned}
\pi(\beta, \tau \mid z, y) &\propto \pi(\beta, \tau, z \mid y) \\
&\propto \pi(\beta, \tau) \cdot f(y, z \mid \beta, \tau) \\
&\propto \tau^{\frac{k}{2}} \exp\left\{-\frac{\beta^{\mathrm{T}}\tau C\beta}{2} + \beta^{\mathrm{T}}\tau Ca\right\} \cdot \tau^{p-1} \exp\left\{-\left(q + \frac{a^{\mathrm{T}}Ca}{2}\right)\tau\right\} \\
&\quad \times \exp\left\{-\frac{\beta^{\mathrm{T}}\tau X^{\mathrm{T}}ZX\beta}{2} + \beta^{\mathrm{T}}\tau X^{\mathrm{T}}Zy\right\} \cdot \tau^{\frac{n}{2}} \exp\left\{-\frac{y^{\mathrm{T}}Zy}{2}\tau\right\} \\
&= \tau^{\frac{k}{2}} \exp\left\{-\frac{1}{2}\beta^{\mathrm{T}}\tau \underbrace{\left(C + X^{\mathrm{T}}ZX\right)}_{C_n}\beta + \beta^{\mathrm{T}}\tau\left(Ca + X^{\mathrm{T}}Zy\right)\right\} \\
&\quad \times \tau^{p+\frac{n}{2}-1} \exp\left\{-\left(q + \frac{a^{\mathrm{T}}Ca + y^{\mathrm{T}}Zy}{2}\right)\tau\right\} \\
&= \tau^{\frac{k}{2}} \exp\left\{-\frac{\beta^{\mathrm{T}}\tau C_n\beta}{2} + \beta^{\mathrm{T}}\tau \underbrace{\left(C + X^{\mathrm{T}}ZX\right)}_{C_n} \underbrace{\left(C + X^{\mathrm{T}}ZX\right)^{-1}\left(Ca + X^{\mathrm{T}}Zy\right)}_{a_n}\right\} \\
&\quad \times \tau^{p+\frac{n}{2}-1} \exp\left\{-\left(q + \frac{a^{\mathrm{T}}Ca + y^{\mathrm{T}}Zy}{2}\right)\tau\right\} \\
&= \tau^{\frac{k}{2}} \exp\left\{-\frac{\beta^{\mathrm{T}}\tau C_n\beta}{2} + \beta^{\mathrm{T}}\tau C_n a_n - \frac{a_n^{\mathrm{T}}\tau C_n a_n}{2} + \frac{a_n^{\mathrm{T}}\tau C_n a_n}{2}\right\} \\
&\quad \times \tau^{p+\frac{n}{2}-1} \exp\left\{-\left(q + \frac{a^{\mathrm{T}}Ca + y^{\mathrm{T}}Zy}{2}\right)\tau\right\} \\
&= \tau^{\frac{k}{2}} \exp\left\{-\frac{\tau(\beta - a_n)^{\mathrm{T}}C_n(\beta - a_n)}{2}\right\} \\
&\quad \times \tau^{p+\frac{n}{2}-1} \exp\left\{-\left(q + \frac{a^{\mathrm{T}}Ca + y^{\mathrm{T}}Zy - a_n^{\mathrm{T}}C_n a_n}{2}\right)\tau\right\}.
\end{aligned}
$$

We calculate that:

$$a_n^{\mathrm{T}} C_n a_n = \left(Ca + X^{\mathrm{T}} Zy\right)^{\mathrm{T}} \left(C + X^{\mathrm{T}} ZX\right)^{-1} \left(Ca + X^{\mathrm{T}} Zy\right).$$

In other words,

$$\beta \mid \tau, z, y \sim \mathcal{N}_k\left(\left(C + X^{\mathrm{T}} ZX\right)^{-1} \left(Ca + X^{\mathrm{T}} Zy\right), \tau^{-1} \left(C + X^{\mathrm{T}} ZX\right)^{-1}\right),$$

$$\tau \mid z, y \sim \mathrm{Gamma}\left(p + \frac{n}{2}, q + \frac{a^{\mathrm{T}} Ca + y^{\mathrm{T}} Zy - a_n^{\mathrm{T}} C_n a_n}{2}\right).$$

Furthermore, we get the conditional posterior distribution of τ as follows:

$$\begin{aligned}
\pi(\tau \mid \beta, z, y) &\propto \pi(\beta, \tau) \cdot f(y, z \mid \beta, \tau) \\
&\propto \tau^{p + \frac{k}{2} - 1} \exp\left\{-\left[q + \frac{(\beta - a)^{\mathrm{T}} C(\beta - a)}{2}\right]\tau\right\} \cdot \tau^{\frac{n}{2}} \exp\left\{-\frac{(y - X\beta)^{\mathrm{T}} Z(y - X\beta)}{2}\tau\right\} \\
&= \tau^{p + \frac{n+k}{2} - 1} \exp\left\{-\left[q + \frac{(\beta - a)^{\mathrm{T}} C(\beta - a) + (y - X\beta)^{\mathrm{T}} Z(y - X\beta)}{2}\right]\tau\right\}.
\end{aligned}$$

In other words,

$$\tau \mid \beta, z, y \sim \mathrm{Gamma}\left(p + \frac{n+k}{2}, q + \frac{(\beta - a)^{\mathrm{T}} C(\beta - a) + (y - X\beta)^{\mathrm{T}} Z(y - X\beta)}{2}\right).$$

Finally, we get the conditional posterior distribution of the latent variables z_i as follows:

$$f(z_i \mid y_i, \beta, \tau) \propto f(y_i, z_i \mid \beta, \tau) \propto z_i^{\frac{\nu+1}{2} - 1} \exp\left\{-\frac{\nu + \tau\left(y_i - x_i^{\mathrm{T}}\beta\right)^2}{2} z_i\right\}.$$

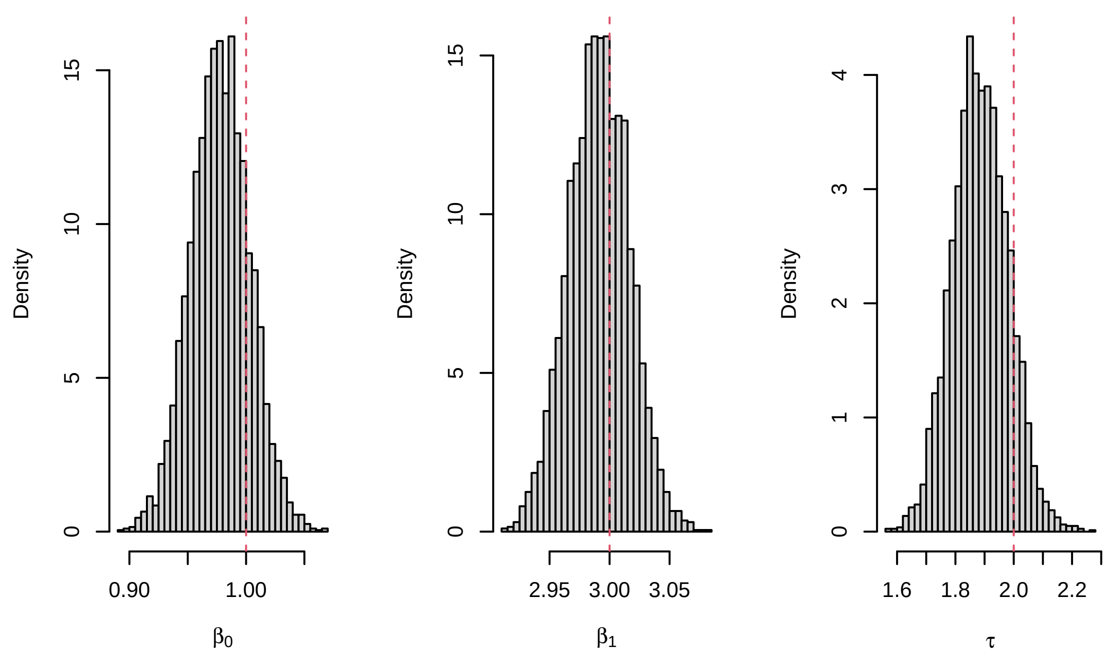
In other words,

$$z_i \mid y_i, \beta, \tau \sim \mathrm{Gamma}\left(\frac{\nu+1}{2}, \frac{\nu + \tau\left(y_i - x_i^{\mathrm{T}}\beta\right)^2}{2}\right).$$

First, we implement a Gibbs sampler which alternately simulates from the conditional posterior distributions of the parameters β, τ and the latent variables z_i.

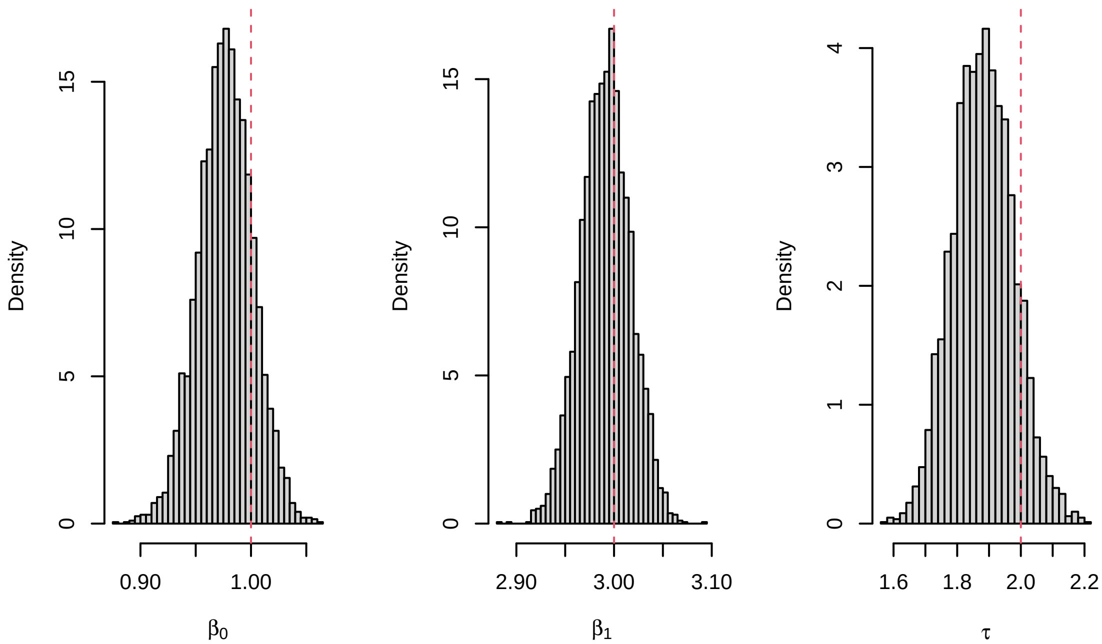
```
MCMCtlm = function(Y, X, beta0, tau0, nu, a, C, p, q, niter, nburn) {
    library(MASS)
    n = length(Y)
    k = dim(X)[2]
    beta = matrix(0, niter, k)
    tau = numeric(niter)
    Z = matrix(0, niter, n)
    beta[1, ] = beta0
    tau[1] = tau0
    Z[1, ] = rgamma(n, (nu + 1)/2, (nu + tau[1] * (Y - X %*% beta[1, ])^2)/2)
    for (i in 2:niter) {
        Cn = C + crossprod(X, Z[i - 1, ] * X)
        an = solve(Cn, C %*% a + crossprod(X, Z[i - 1, ] * Y))
        beta[i, ] = mvrnorm(1, an, solve(Cn)/tau[i - 1])
```

```
        tau[i] = rgamma(1, p + (n + k)/2, q + (crossprod(beta[i, ] - a, C %*%
            (beta[i, ] - a)) + crossprod(Y - X %*% beta[i, ], Z[i - 1, ] * (Y -
            X %*% beta[i, ])))/2)
        Z[i, ] = rgamma(n, (nu + 1)/2, (nu + tau[i] * (Y - X %*% beta[i, ])^2)/2)
    }
    return(list(beta = beta[-(1:nburn), ], tau = tau[-(1:nburn)], Z = Z[-(1:nburn),
        ]))
}

library(mvtnorm)
n = 1000
k = 2
beta = c(1, 3)
tau = 2
nu = 10
X = cbind(1, rnorm(n))
Y = X %*% beta + rmvt(n, matrix(tau^(-1)), nu)
posterior = MCMCtlm(Y, X, numeric(k), 1, nu, numeric(k), matrix(0, k, k), 0.5,
    0, 5000, 1000)
par(mfrow = c(1, 3))
hist(posterior$beta[, 1], "FD", freq = FALSE, main = NA, xlab = expression(beta[0]))
abline(v = beta[1], col = 2, lty = 2)
hist(posterior$beta[, 2], "FD", freq = FALSE, main = NA, xlab = expression(beta[1]))
abline(v = beta[2], col = 2, lty = 2)
hist(posterior$tau, "FD", freq = FALSE, main = NA, xlab = expression(tau))
abline(v = tau, col = 2, lty = 2)
```



Next, we implement a Gibbs sampler which alternately simulates from the conditional posterior distributions of the parameter (β, τ) and the latent variables z_i.

```
MCMCtlm = function(Y, X, beta0, tau0, nu, a, C, p, q, niter, nburn) {
    library(MASS)
    n = length(Y)
    k = dim(X)[2]
    beta = matrix(0, niter, k)
    tau = numeric(niter)
    Z = matrix(0, niter, n)
    beta[1, ] = beta0
    tau[1] = tau0
    Z[1, ] = rgamma(n, (nu + 1)/2, (nu + tau[1] * (Y - X %*% beta[1, ])^2)/2)
    for (i in 2:niter) {
        Cn = C + crossprod(X, Z[i - 1, ] * X)
        an = solve(Cn, C %*% a + crossprod(X, Z[i - 1, ] * Y))
        qn = q + (crossprod(a, C %*% a) + crossprod(Y, Z[i - 1, ] * Y) - crossprod(C %*%
            a + crossprod(X, Z[i - 1, ] * Y), an))/2
        tau[i] = rgamma(1, p + n/2, qn)
        beta[i, ] = mvrnorm(1, an, solve(Cn)/tau[i])
        Z[i, ] = rgamma(n, (nu + 1)/2, (nu + tau[i] * (Y - X %*% beta[i, ])^2)/2)
    }
    return(list(beta = beta[-(1:nburn), ], tau = tau[-(1:nburn)], Z = Z[-(1:nburn),
        ]))
}

posterior = MCMCtlm(Y, X, numeric(k), 1, nu, numeric(k), matrix(0, k, k), 0.5,
    0, 5000, 1000)
par(mfrow = c(1, 3))
hist(posterior$beta[, 1], "FD", freq = FALSE, main = NA, xlab = expression(beta[0]))
abline(v = beta[1], col = 2, lty = 2)
hist(posterior$beta[, 2], "FD", freq = FALSE, main = NA, xlab = expression(beta[1]))
abline(v = beta[2], col = 2, lty = 2)
hist(posterior$tau, "FD", freq = FALSE, main = NA, xlab = expression(tau))
abline(v = tau, col = 2, lty = 2)
```



1.5 Multivariate Normal Distribution

Definition 1.2. We define the multivariate Gamma function as follows:

$$\Gamma_k(x) = \pi^{\frac{k(k-1)}{4}} \prod_{j=1}^{k} \Gamma\left(x + \frac{1-j}{2}\right), \quad x > \frac{k-1}{2}, \quad k \in \mathbb{N}.$$

Definition 1.3. We say that a positive definite random matrix $X \in \mathbb{R}^{k\times k}$ follows the Wishart distribution with positive definite scale matrix $A \in \mathbb{R}^{k\times k}$ and $\nu > 0$ degrees of freedom, i.e. $X \sim \mathcal{W}_k(A, \nu)$, if it has the following probability density function:

$$f_X(x \mid A, \nu) = \frac{1}{2^{\frac{\nu k}{2}} |A|^{\frac{\nu}{2}} \Gamma_k\left(\frac{\nu}{2}\right)} |x|^{\frac{\nu-k-1}{2}} e^{-\frac{1}{2}\mathrm{tr}\left(A^{-1}x\right)}, \quad x \in \mathbb{R}^{k\times k}.$$

Let $y_1, \ldots, y_n$ be a random sample from the multivariate normal distribution $\mathcal{N}_k(\mu, \Sigma)$.

a. Consider prior independence with prior distributions $\mu \sim \mathcal{N}_k\left(a, C^{-1}\right)$ and $\Omega \sim \mathcal{W}_k\left(A^{-1}, d\right)$. Calculate the conditional posterior distributions of μ and Ω.

b. Consider the conjugate prior distribution $\mu \mid \Omega \sim \mathcal{N}_k\left(a, c^{-1}\Omega^{-1}\right)$, $\Omega \sim \mathcal{W}_k\left(A^{-1}, d\right)$. Calculate the conditional and marginal posterior distributions of μ and Ω.

Solution.

a. The joint prior distribution may be written as follows:

$$\begin{aligned}\pi(\mu, \Omega) &= \pi(\mu) \cdot \pi(\Omega) \\ &= (2\pi)^{-\frac{k}{2}} \left|C^{-1}\right|^{-\frac{1}{2}} \exp\left\{-\frac{(\mu-a)^{\mathrm{T}} C(\mu-a)}{2}\right\} \cdot \frac{|A|^{\frac{d}{2}}}{2^{\frac{dk}{2}} \Gamma_k\left(\frac{d}{2}\right)} |\Omega|^{\frac{d-k-1}{2}} e^{-\frac{1}{2}\mathrm{tr}(A\Omega)} \\ &\propto \exp\left\{-\frac{\mu^{\mathrm{T}} C\mu - 2\mu^{\mathrm{T}} Ca + a^{\mathrm{T}} Ca}{2}\right\} \cdot |\Omega|^{\frac{d-k-1}{2}} e^{-\frac{1}{2}\mathrm{tr}(A\Omega)}\end{aligned}$$

$$\propto \exp\left\{-\frac{\mu^{\mathrm{T}}C\mu}{2}+\mu^{\mathrm{T}}Ca\right\}\cdot|\Omega|^{\frac{d-k-1}{2}}e^{-\frac{1}{2}\mathrm{tr}(A\Omega)}.$$

Lemma 1.1. Let $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. Then,

$$x^{\mathrm{T}}Ax = \mathrm{tr}\left(x^{\mathrm{T}}Ax\right) = \mathrm{tr}\left(xx^{\mathrm{T}}A\right).$$

The likelihood of the sample is given by:

$$\begin{aligned}
f(y \mid \mu, \Omega) &= \prod_{i=1}^{n} f(y_i \mid \mu, \Omega) \\
&= \prod_{i=1}^{n} (2\pi)^{-\frac{k}{2}}\left|\Omega^{-1}\right|^{-\frac{1}{2}} \exp\left\{-\frac{(y_i-\mu)^{\mathrm{T}}\Omega(y_i-\mu)}{2}\right\} \\
&\propto |\Omega|^{\frac{n}{2}} \exp\left\{-\frac{1}{2}\mathrm{tr}\left[\sum_{i=1}^{n}(y_i-\mu)(y_i-\mu)^{\mathrm{T}}\Omega\right]\right\} \\
&= |\Omega|^{\frac{n}{2}} \exp\left\{-\sum_{i=1}^{n}\frac{y_i^{\mathrm{T}}\Omega y_i - 2\mu^{\mathrm{T}}\Omega y_i + \mu^{\mathrm{T}}\Omega\mu}{2}\right\} \\
&= \exp\left\{-\frac{\mu^{\mathrm{T}}n\Omega\mu}{2}+\mu^{\mathrm{T}}n\Omega\overline{y}\right\}\cdot|\Omega|^{\frac{n}{2}}\exp\left\{-\frac{1}{2}\mathrm{tr}\left(\sum_{i=1}^{n}y_iy_i^{\mathrm{T}}\Omega\right)\right\}.
\end{aligned}$$

Therefore, we get the conditional posterior distributions of μ and Ω as follows:

$$\begin{aligned}
\pi(\mu \mid \Omega, y) &\propto \pi(\mu, \Omega \mid y) \\
&\propto \pi(\mu, \Omega)\cdot f(y \mid \mu, \Omega) \\
&\propto \exp\left\{-\frac{\mu^{\mathrm{T}}C\mu}{2}+\mu^{\mathrm{T}}Ca\right\}\cdot\exp\left\{-\frac{\mu^{\mathrm{T}}n\Omega\mu}{2}+\mu^{\mathrm{T}}n\Omega\overline{y}\right\} \\
&= \exp\left\{-\frac{1}{2}\mu^{\mathrm{T}}\underbrace{(C+n\Omega)}_{C_n}\mu+\mu^{\mathrm{T}}\left(Ca+n\Omega\overline{y}\right)\right\} \\
&= \exp\left\{-\frac{\mu^{\mathrm{T}}\left(C+n\Omega\right)\mu}{2}+\mu^{\mathrm{T}}\underbrace{(C+n\Omega)}_{C_n}\underbrace{(C+n\Omega)^{-1}\left(Ca+n\Omega\overline{y}\right)}_{a_n}\right\},
\end{aligned}$$

$$\begin{aligned}
\pi(\Omega \mid \mu, y) &\propto \pi(\mu, \Omega)\cdot f(y \mid \mu, \Omega) \\
&\propto |\Omega|^{\frac{d-k-1}{2}}e^{-\frac{1}{2}\mathrm{tr}(A\Omega)}\cdot|\Omega|^{\frac{n}{2}}\exp\left\{-\frac{1}{2}\mathrm{tr}\left[\sum_{i=1}^{n}(y_i-\mu)(y_i-\mu)^{\mathrm{T}}\Omega\right]\right\} \\
&= |\Omega|^{\frac{d+n-k-1}{2}}\exp\left\{-\frac{1}{2}\mathrm{tr}\left[\left(A+\sum_{i=1}^{n}(y_i-\mu)(y_i-\mu)^{\mathrm{T}}\right)\Omega\right]\right\}.
\end{aligned}$$

In other words,

$$\mu \mid \Omega, y \sim \mathcal{N}_k\left((C+n\Omega)^{-1}\left(Ca+n\Omega\overline{y}\right), (C+n\Omega)^{-1}\right),$$

$$\Omega \mid \mu, y \sim \mathcal{W}_k\left(\left(A+\sum_{i=1}^{n}(y_i-\mu)(y_i-\mu)^{\mathrm{T}}\right)^{-1}, d+n\right).$$

Definition 1.4. We define the Kronecker product of two matrices $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times q}$ as the following matrix:

$$C = A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nm}B \end{bmatrix} \in \mathbb{R}^{np\times mq}.$$

Lemma 1.2. Let $A \in \mathbb{R}^{n\times n}$ be a positive definite matrix, $x, a \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then, it follows that:

$$\frac{\partial c}{\partial x} = \mathbf{0}_n, \quad \frac{\partial a}{\partial x} = \mathbf{0}_{n\times n}, \quad \frac{\partial Ax}{\partial x} = A, \quad \frac{\partial a^{\mathrm{T}}x}{\partial x} = a, \quad \frac{\partial x^{\mathrm{T}}Ax}{\partial x} = 2Ax,$$

$$\frac{\partial aa^{\mathrm{T}}}{\partial x} = \mathbf{0}_{n^2\times n}, \quad \frac{\partial ax^{\mathrm{T}}}{\partial x} = \frac{\partial xa^{\mathrm{T}}}{\partial x} = a \otimes I_n, \quad \frac{\partial xx^{\mathrm{T}}}{\partial x} = x \otimes I_n + I_n \otimes x,$$

$$\frac{\partial x^{\mathrm{T}}Ax}{\partial A} = xx^{\mathrm{T}}, \quad \frac{\partial \log|A|}{\partial A} = A^{-1}, \quad \frac{\partial A^{-1}}{\partial A} = -A^{-1} \otimes A^{-1}, \quad \frac{\partial Ax}{\partial A} = x^{\mathrm{T}} \otimes I_n.$$

Lemma 1.3. Let $X \in \mathbb{R}^k$ be a random vector and $A \in \mathbb{R}^{n\times m}$ a constant matrix. Then, it follows that:

$$\mathbb{E}\left(A \otimes X\right) = A \otimes \mathbb{E}(X), \quad \mathbb{E}\left(X \otimes A\right) = \mathbb{E}(X) \otimes A.$$

Lemma 1.4. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$. Then, it follows that:

$$|A \otimes B| = |A|^m |B|^n.$$

We can calculate Jeffreys' prior for the multivariate normal distribution as follows:

$$\log f(y \mid \mu, \Omega) = \frac{1}{2}\log|\Omega| - \frac{k}{2}\log(2\pi) - \frac{(y-\mu)^{\mathrm{T}}\Omega(y-\mu)}{2},$$

$$\frac{\partial \log f(y \mid \mu, \Omega)}{\partial \mu} = \Omega(y-\mu) \in \mathbb{R}^k, \quad \frac{\partial \log f(y \mid \mu, \Omega)}{\partial \Omega} = \frac{1}{2}\Omega^{-1} - \frac{(y-\mu)(y-\mu)^{\mathrm{T}}}{2} \in \mathbb{R}^{k\times k},$$

$$\frac{\partial^2 \log f(y \mid \mu, \Omega)}{\partial \mu \partial \mu} = -\Omega \in \mathbb{R}^{k\times k}, \quad \frac{\partial^2 \log f(y \mid \mu, \Omega)}{\partial \Omega \partial \Omega} = -\frac{1}{2}\Omega^{-1} \otimes \Omega^{-1} \in \mathbb{R}^{k^2\times k^2},$$

$$\frac{\partial^2 \log f(y \mid \mu, \Omega)}{\partial \Omega \partial \mu} = (y-\mu)^{\mathrm{T}} \otimes I_k \in \mathbb{R}^{k\times k^2}, \quad \frac{\partial^2 \log f(y \mid \mu, \Omega)}{\partial \mu \partial \Omega} = \frac{(y-\mu) \otimes I_k + I_k \otimes (y-\mu)}{2} \in \mathbb{R}^{k^2\times k},$$

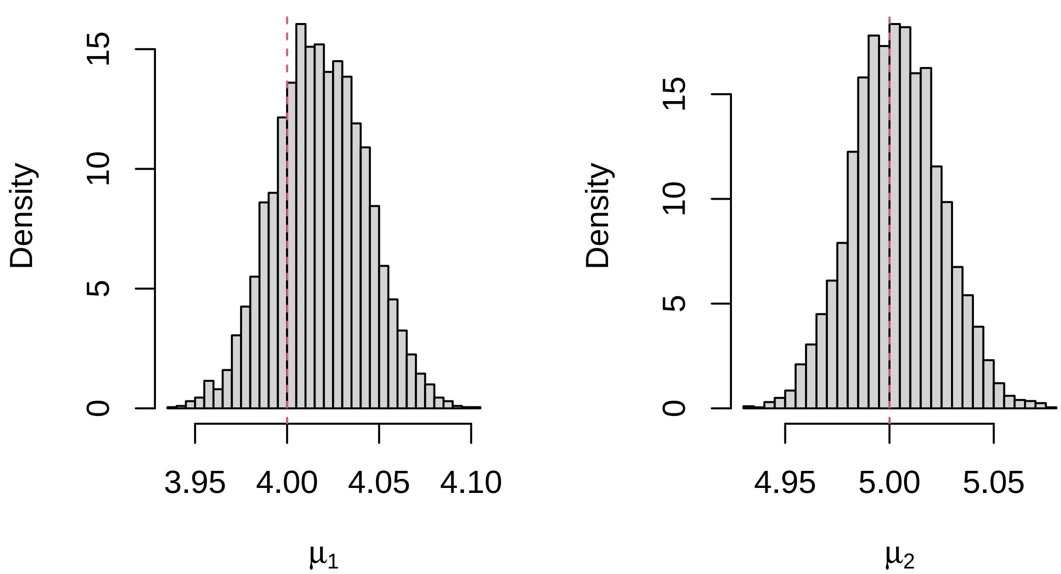
$$\mathcal{I}(\mu, \Omega) = \mathbb{E}\left[-\frac{\partial^2 \log f(y \mid \mu, \Omega)}{\partial(\mu, \Omega)\partial(\mu, \Omega)}\right] = \begin{bmatrix} \Omega & \mathbf{0}_{k\times k^2} \\ \mathbf{0}_{k^2\times k} & \frac{1}{2}\Omega^{-1} \otimes \Omega^{-1} \end{bmatrix},$$

$$J(\mu, \Omega) \propto \sqrt{|\mathcal{I}(\mu, \Omega)|} = \sqrt{|\Omega| \left|\frac{1}{2}\Omega^{-1} \otimes \Omega^{-1}\right|} \propto \sqrt{|\Omega|\,|\Omega|^{-k}\,|\Omega|^{-k}} = |\Omega|^{\frac{1-2k}{2}}.$$

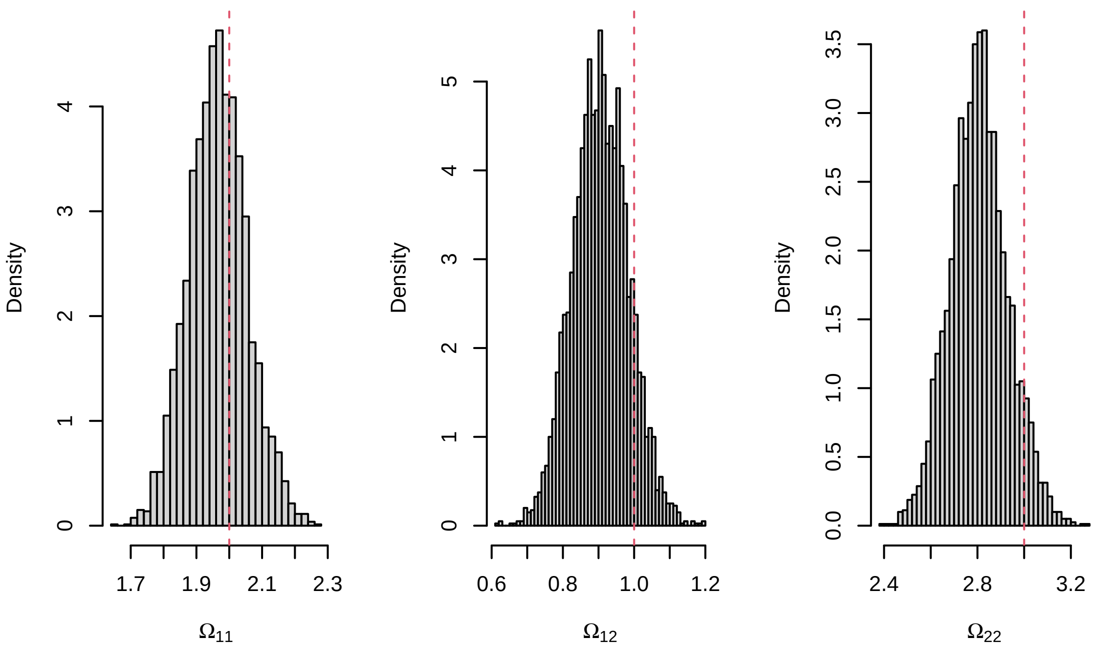
We observe that the improper Jeffreys' prior results for $a = \mathbf{0}_k$, $C = A = \mathbf{0}_{k\times k}$ and $d = 2 - k$

```
MCMCmvnorm = function(Y, mu0, Omega0, a, C, A, d, niter, nburn) {
    library(MASS)
    n = dim(Y)[1]
```

```
    k = dim(Y)[2]
    S = colSums(Y)
    mu = matrix(0, niter, k)
    Omega = array(0, c(k, k, niter))
    mu[1, ] = mu0
    Omega[, , 1] = Omega0
    for (i in 2:niter) {
        Cn = C + n * Omega[, , i - 1]
        an = solve(Cn, C %*% a + Omega[, , i - 1] %*% S)
        mu[i, ] = mvrnorm(1, an, solve(Cn))
        Omega[, , i] = rWishart(1, d + n, solve(A + tcrossprod(t(Y) - mu[i,
            ])))
    }
    return(list(mu = mu[-(1:nburn), ], Omega = Omega[, , -(1:nburn)]))
}

library(MASS)
n = 1000
k = 2
mu = c(4, 5)
Omega = matrix(c(2, 1, 1, 3), k)
Y = mvrnorm(n, mu, solve(Omega))
posterior = MCMCmvnorm(Y, numeric(k), diag(k), numeric(k), matrix(0, k, k),
    matrix(0, k, k), 2 - k, 5000, 1000)
par(mfrow = c(1, 2))
hist(posterior$mu[, 1], "FD", freq = FALSE, main = NA, xlab = expression(mu[1]))
abline(v = mu[1], col = 2, lty = 2)
hist(posterior$mu[, 2], "FD", freq = FALSE, main = NA, xlab = expression(mu[2]))
abline(v = mu[2], col = 2, lty = 2)
```



```r
par(mfrow = c(1, 3))
hist(posterior$Omega[1, 1, ], "FD", freq = FALSE, main = NA, xlab = expression(Omega[11]))
abline(v = Omega[1, 1], col = 2, lty = 2)
hist(posterior$Omega[1, 2, ], "FD", freq = FALSE, main = NA, xlab = expression(Omega[12]))
abline(v = Omega[1, 2], col = 2, lty = 2)
hist(posterior$Omega[2, 2, ], "FD", freq = FALSE, main = NA, xlab = expression(Omega[22]))
abline(v = Omega[2, 2], col = 2, lty = 2)
```



b. The joint prior distribution may be written as follows:

$$
\begin{aligned}
\pi(\mu, \Omega) &= \pi(\mu \mid \Omega) \cdot \pi(\Omega) \\
&= (2\pi)^{-\frac{k}{2}} \left|c^{-1}\Omega^{-1}\right|^{-\frac{1}{2}} \exp\left\{-\frac{c(\mu-a)^{\mathrm{T}}\Omega(\mu-a)}{2}\right\} \cdot \frac{|A|^{\frac{d}{2}}}{2^{\frac{dk}{2}}\Gamma_k\left(\frac{d}{2}\right)} |\Omega|^{\frac{d-k-1}{2}} e^{-\frac{1}{2}\mathrm{tr}(A\Omega)} \\
&\propto |\Omega|^{\frac{d+1-k-1}{2}} \exp\left\{-\frac{1}{2}\mathrm{tr}\left[\left(A + c(\mu-a)(\mu-a)^{\mathrm{T}}\right)\Omega\right]\right\} \\
&= |\Omega|^{\frac{1}{2}} \exp\left\{-c\frac{\mu^{\mathrm{T}}\Omega\mu - 2\mu^{\mathrm{T}}\Omega a + a^{\mathrm{T}}\Omega a}{2}\right\} \cdot |\Omega|^{\frac{d-k-1}{2}} e^{-\frac{1}{2}\mathrm{tr}(A\Omega)} \\
&= |\Omega|^{\frac{1}{2}} \exp\left\{-\frac{\mu^{\mathrm{T}}c\Omega\mu}{2} + \mu^{\mathrm{T}}c\Omega a\right\} \cdot |\Omega|^{\frac{d-k-1}{2}} \exp\left\{-\frac{1}{2}\mathrm{tr}\left[\left(A + caa^{\mathrm{T}}\right)\Omega\right]\right\}.
\end{aligned}
$$

Therefore, we get the joint posterior distribution of μ and Ω as follows:

$$
\begin{aligned}
\pi(\mu, \Omega \mid y) &\propto \pi(\mu, \Omega) \cdot f(y \mid \mu, \Omega) \\
&\propto |\Omega|^{\frac{1}{2}} \exp\left\{-\frac{\mu^{\mathrm{T}}c\Omega\mu}{2} + \mu^{\mathrm{T}}c\Omega a\right\} \cdot |\Omega|^{\frac{d-k-1}{2}} \exp\left\{-\frac{1}{2}\mathrm{tr}\left[\left(A + caa^{\mathrm{T}}\right)\Omega\right]\right\} \\
&\quad \times \exp\left\{-\frac{\mu^{\mathrm{T}}n\Omega\mu}{2} + \mu^{\mathrm{T}}n\Omega\overline{y}\right\} \cdot |\Omega|^{\frac{n}{2}} \exp\left\{-\frac{1}{2}\mathrm{tr}\left(\sum_{i=1}^{n} y_i y_i^{\mathrm{T}}\Omega\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= |\Omega|^{\frac{1}{2}} \exp\left\{ -\frac{1}{2}\mu^{\mathrm{T}} \underbrace{(c+n)}_{c_n} \Omega\mu + \mu^{\mathrm{T}}\Omega\left(ca+n\overline{y}\right) \right\} \\
&\quad \times |\Omega|^{\frac{d+n-k-1}{2}} \exp\left\{ -\frac{1}{2}\mathrm{tr}\left[\left(A + caa^{\mathrm{T}} + \sum_{i=1}^{n} y_i y_i^{\mathrm{T}} \right) \Omega \right] \right\} \\
&= |\Omega|^{\frac{1}{2}} \exp\left\{ -\frac{\mu^{\mathrm{T}}(c+n)\Omega\mu}{2} + \mu^{\mathrm{T}} \underbrace{(c+n)}_{c_n} \Omega \underbrace{\frac{ca+n\overline{y}}{c+n}}_{a_n} \right\} \\
&\quad \times |\Omega|^{\frac{d+n-k-1}{2}} \exp\left\{ -\frac{1}{2}\mathrm{tr}\left[\left(A + caa^{\mathrm{T}} + \sum_{i=1}^{n} y_i y_i^{\mathrm{T}} \right) \Omega \right] \right\} \\
&= |\Omega|^{\frac{1}{2}} \exp\left\{ -\frac{\mu^{\mathrm{T}}c_n\Omega\mu}{2} + \mu^{\mathrm{T}}c_n\Omega a_n - \frac{a_n^{\mathrm{T}}c_n\Omega a_n}{2} + \frac{a_n^{\mathrm{T}}c_n\Omega a_n}{2} \right\} \\
&\quad \times |\Omega|^{\frac{d+n-k-1}{2}} \exp\left\{ -\frac{1}{2}\mathrm{tr}\left[\left(A + caa^{\mathrm{T}} + \sum_{i=1}^{n} y_i y_i^{\mathrm{T}} \right) \Omega \right] \right\} \\
&= |\Omega|^{\frac{1}{2}} \exp\left\{ -\frac{c_n(\mu-a_n)^{\mathrm{T}}\Omega(\mu-a_n)}{2} \right\} \\
&\quad \times |\Omega|^{\frac{d+n-k-1}{2}} \exp\left\{ -\frac{1}{2}\mathrm{tr}\left[\left(A + caa^{\mathrm{T}} + \sum_{i=1}^{n} y_i y_i^{\mathrm{T}} - c_n a_n a_n^{\mathrm{T}} \right) \Omega \right] \right\}.
\end{aligned}
$$

We calculate that:

$$
c_n a_n a_n^{\mathrm{T}} = \frac{\left(ca+n\overline{y}\right)\left(ca+n\overline{y}\right)^{\mathrm{T}}}{c+n}.
$$

In other words,

$$
\mu \mid \Omega, y \sim \mathcal{N}_k\left(\frac{ca+n\overline{y}}{c+n}, \frac{1}{c+n}\Omega^{-1} \right),
$$

$$
\Omega \mid y \sim \mathcal{W}_k\left(\left(A + caa^{\mathrm{T}} + \sum_{i=1}^{n} y_i y_i^{\mathrm{T}} - \frac{\left(ca+n\overline{y}\right)\left(ca+n\overline{y}\right)^{\mathrm{T}}}{c+n} \right)^{-1}, d+n \right).
$$

Furthermore, we get the conditional posterior distribution of Ω as follows:

$$
\begin{aligned}
\pi(\Omega \mid \mu, y) &\propto \pi(\mu, \Omega) \cdot f(y \mid \mu, \Omega) \\
&\propto |\Omega|^{\frac{d+1-k-1}{2}} \exp\left\{ -\frac{1}{2}\mathrm{tr}\left[\left(A + c(\mu-a)(\mu-a)^{\mathrm{T}} \right) \Omega \right] \right\} \\
&\quad \times |\Omega|^{\frac{n}{2}} \exp\left\{ -\frac{1}{2}\mathrm{tr}\left[\sum_{i=1}^{n} (y_i-\mu)(y_i-\mu)^{\mathrm{T}}\Omega \right] \right\} \\
&= |\Omega|^{\frac{d+n+1-k-1}{2}} \exp\left\{ -\frac{1}{2}\mathrm{tr}\left[\left(A + c(\mu-a)(\mu-a)^{\mathrm{T}} + \sum_{i=1}^{n} (y_i-\mu)(y_i-\mu)^{\mathrm{T}} \right) \Omega \right] \right\}.
\end{aligned}
$$

In other words,

$$
\Omega \mid \mu, y \sim \mathcal{W}_k\left(\left(A + c(\mu-a)(\mu-a)^{\mathrm{T}} + \sum_{i=1}^{n} (y_i-\mu)(y_i-\mu)^{\mathrm{T}} \right)^{-1}, d+n+1 \right).
$$

Definition 1.5. We say that a random vector $X \in \mathbb{R}^k$ follows the multivariate Student's t distribution with mean vector $\mu \in \mathbb{R}^k$, positive definite covariance matrix $\Sigma \in \mathbb{R}^{k\times k}$ and $\nu > 0$ degrees of freedom, i.e. $X \sim t_\nu(\mu, \Sigma)$, if it has the following probability density function:

$$f_X(x \mid \mu, \Sigma, \nu) = \frac{\Gamma\left(\frac{\nu+k}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{(\nu\pi)^{\frac{k}{2}}|\Sigma|^{\frac{1}{2}}} \left[1 + \frac{1}{\nu}(x-\mu)^{\mathrm{T}}\Sigma^{-1}(x-\mu)\right]^{-\frac{\nu+k}{2}}, \quad x \in \mathbb{R}^k.$$

Lemma 1.5. Let $x, y \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ an invertible matrix. Then,

$$\left|A + xy^{\mathrm{T}}\right| = |A|\left(1 + y^{\mathrm{T}}A^{-1}x\right).$$

Finally, We define:

$$d_n = d + n - k + 1, \quad A_n = A + caa^{\mathrm{T}} + \sum_{i=1}^n y_i y_i^{\mathrm{T}} - c_n a_n a_n^{\mathrm{T}}.$$

Then, we calculate the marginal posterior distribution of μ as follows:

$$\begin{aligned}
\pi(\mu \mid y) &= \int \pi(\mu, \Omega \mid y) d\Omega \\
&\propto \int |\Omega|^{\frac{d+n+1-k-1}{2}} \exp\left\{-\frac{1}{2}\mathrm{tr}\left[\left(A + c(\mu-a)(\mu-a)^{\mathrm{T}} + \sum_{i=1}^n (y_i-\mu)(y_i-\mu)^{\mathrm{T}}\right)\Omega\right]\right\} d\Omega \\
&\propto \left|A + c(\mu-a)(\mu-a)^{\mathrm{T}} + \sum_{i=1}^n (y_i-\mu)(y_i-\mu)^{\mathrm{T}}\right|^{-\frac{d+n+1}{2}} \\
&= \left|A + caa^{\mathrm{T}} + \sum_{i=1}^n y_i y_i^{\mathrm{T}} + (c+n)\mu\mu^{\mathrm{T}} - (ca + n\overline{y})\mu^{\mathrm{T}} - \mu(ca + n\overline{y})^{\mathrm{T}}\right|^{-\frac{d+n-k+1+k}{2}} \\
&= \left|A + caa^{\mathrm{T}} + \sum_{i=1}^n y_i y_i^{\mathrm{T}} - c_n a_n a_n^{\mathrm{T}} + c_n(\mu - a_n)(\mu - a_n)^{\mathrm{T}}\right|^{-\frac{d_n+k}{2}} \\
&\propto \left[1 + c_n(\mu - a_n)^{\mathrm{T}} A_n^{-1}(\mu - a_n)\right]^{-\frac{d_n+k}{2}} \\
&= \left[1 + \frac{1}{d_n}(\mu - a_n)^{\mathrm{T}} c_n d_n A_n^{-1}(\mu - a_n)\right]^{-\frac{d_n+k}{2}}.
\end{aligned}$$

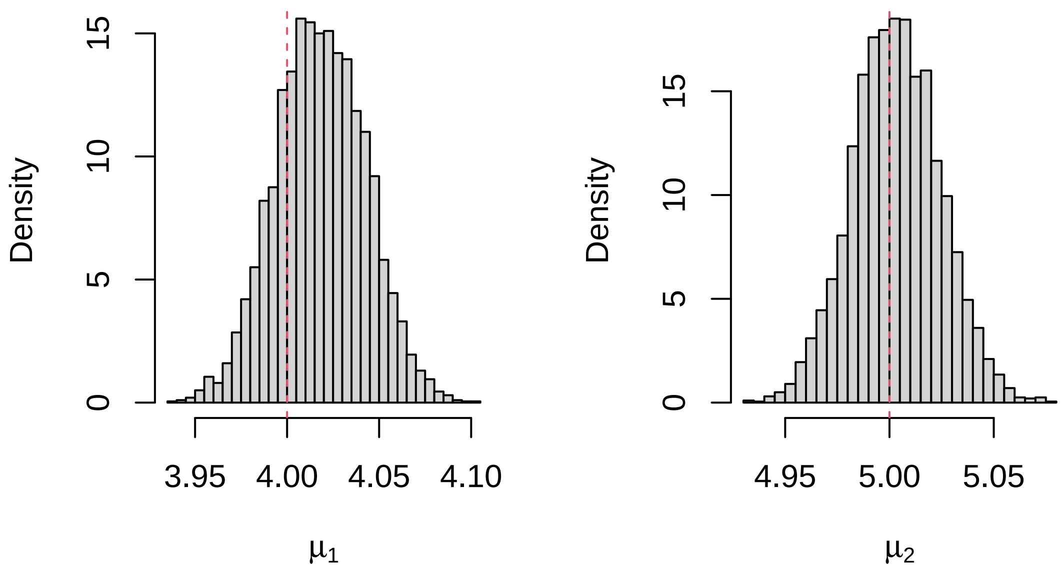
In other words,

$$\mu \mid y \sim t_{d_n}\left(a_n, \frac{1}{c_n d_n}A_n\right).$$

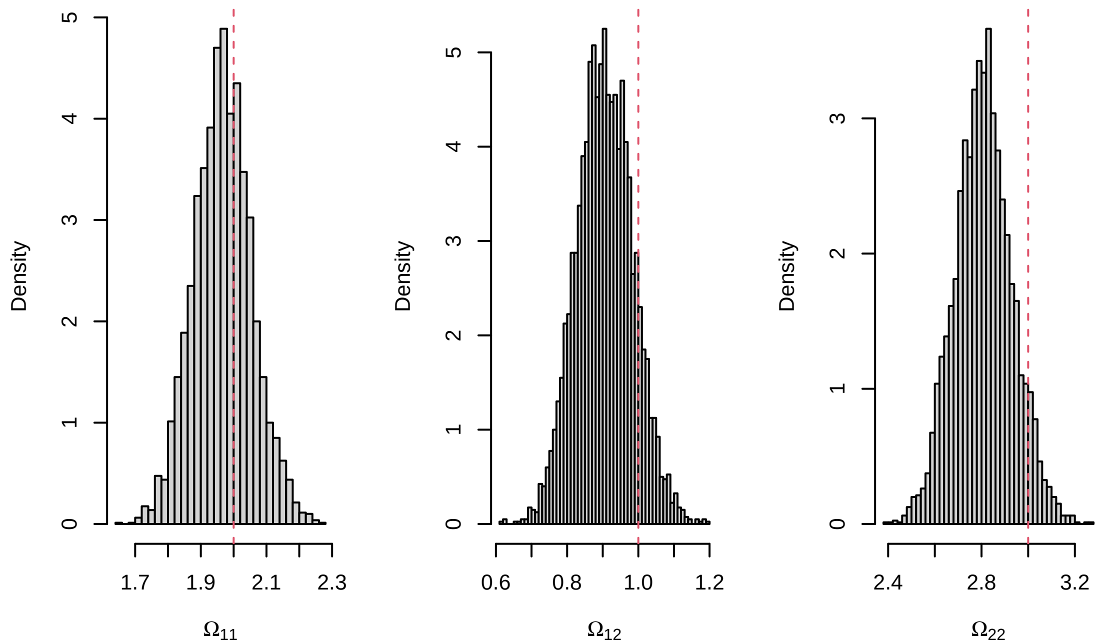
First, we implement a Gibbs sampler which alternately simulates from the conditional posterior distributions of μ and Ω.

```
MCMCmvnorm = function(Y, mu0, Omega0, a, c, A, d, niter, nburn) {
    library(MASS)
    n = dim(Y)[1]
    k = dim(Y)[2]
    S = colSums(Y)
    cn = c + n
    an = (c * a + S)/cn
    mu = matrix(0, niter, k)
```

```
    Omega = array(0, c(k, k, niter))
    mu[1, ] = mu0
    Omega[, , 1] = Omega0
    for (i in 2:niter) {
        mu[i, ] = mvrnorm(1, an, solve(Omega[, , i - 1])/cn)
        Omega[, , i] = rWishart(1, d + n + 1, solve(A + c * tcrossprod(mu[i,
            ] - a) + tcrossprod(t(Y) - mu[i, ])))
    }
    return(list(mu = mu[-(1:nburn), ], Omega = Omega[, , -(1:nburn)]))
}

posterior = MCMCmvnorm(Y, numeric(k), diag(k), numeric(k), 0, matrix(0, k, k),
    2 - k, 5000, 1000)
par(mfrow = c(1, 2))
hist(posterior$mu[, 1], "FD", freq = FALSE, main = NA, xlab = expression(mu[1]))
abline(v = mu[1], col = 2, lty = 2)
hist(posterior$mu[, 2], "FD", freq = FALSE, main = NA, xlab = expression(mu[2]))
abline(v = mu[2], col = 2, lty = 2)
```



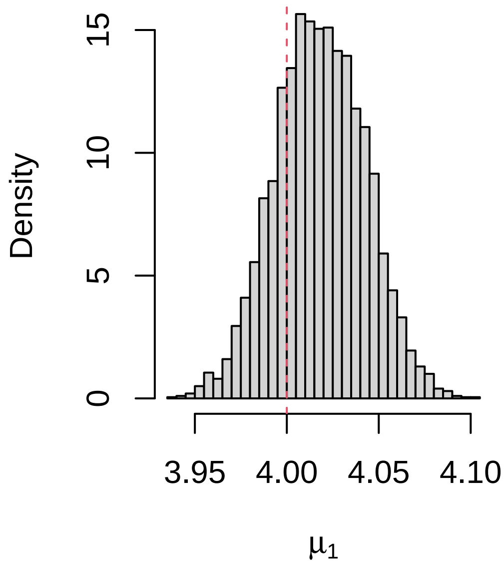

```
par(mfrow = c(1, 3))
hist(posterior$Omega[1, 1, ], "FD", freq = FALSE, main = NA, xlab = expression(Omega[11]))
abline(v = Omega[1, 1], col = 2, lty = 2)
hist(posterior$Omega[1, 2, ], "FD", freq = FALSE, main = NA, xlab = expression(Omega[12]))
abline(v = Omega[1, 2], col = 2, lty = 2)
hist(posterior$Omega[2, 2, ], "FD", freq = FALSE, main = NA, xlab = expression(Omega[22]))
abline(v = Omega[2, 2], col = 2, lty = 2)
```



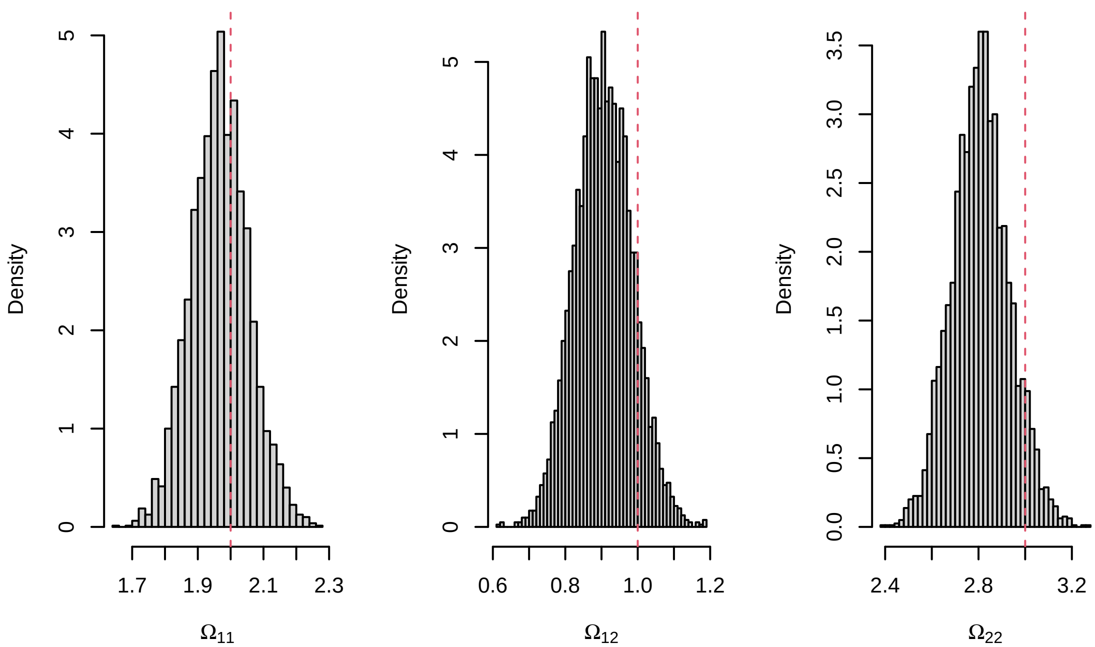
Next, we implement the composition method which first simulates from the marginal posterior distribution of Ω and then from the conditional posterior distribution of μ.

```
CMmvnorm = function(Y, mu0, Omega0, a, c, A, d, niter, nburn) {
    library(MASS)
    n = dim(Y)[1]
    k = dim(Y)[2]
    S = colSums(Y)
    cn = c + n
    an = (c * a + S)/cn
    An = solve(A + c * tcrossprod(a) + crossprod(Y) - cn * tcrossprod(an))
    mu = matrix(0, niter, k)
    Omega = array(0, c(k, k, niter))
    mu[1, ] = mu0
    Omega[, , 1] = Omega0
    for (i in 2:niter) {
        Omega[, , i] = rWishart(1, d + n, An)
        mu[i, ] = mvrnorm(1, an, solve(Omega[, , i])/cn)
    }
    return(list(mu = mu[-(1:nburn), ], Omega = Omega[, , -(1:nburn)]))
}

posterior = CMmvnorm(Y, numeric(k), diag(k), numeric(k), 0, matrix(0, k, k),
    2 - k, 5000, 1000)
par(mfrow = c(1, 2))
hist(posterior$mu[, 1], "FD", freq = FALSE, main = NA, xlab = expression(mu[1]))
abline(v = mu[1], col = 2, lty = 2)
hist(posterior$mu[, 2], "FD", freq = FALSE, main = NA, xlab = expression(mu[2]))
```

```
abline(v = mu[2], col = 2, lty = 2)
```




```
par(mfrow = c(1, 3))
hist(posterior$Omega[1, 1, ], "FD", freq = FALSE, main = NA, xlab = expression(Omega[11]))
abline(v = Omega[1, 1], col = 2, lty = 2)
hist(posterior$Omega[1, 2, ], "FD", freq = FALSE, main = NA, xlab = expression(Omega[12]))
abline(v = Omega[1, 2], col = 2, lty = 2)
hist(posterior$Omega[2, 2, ], "FD", freq = FALSE, main = NA, xlab = expression(Omega[22]))
abline(v = Omega[2, 2], col = 2, lty = 2)
```



1.6 Multivariate Student's t Distribution

Let $y_1, \ldots, y_n$ be a random sample from the multivariate Student's t distribution with mean vector $\mu \in \mathbb{R}^k$, positive definite precision matrix $\Omega \in \mathbb{R}^{k \times k}$ and $\nu > 0$ degrees of freedom, that is:

$$f(y_i \mid \mu, \Omega, \nu) = \frac{\Gamma\left(\frac{\nu+k}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} (\nu\pi)^{-\frac{k}{2}} |\Omega|^{\frac{1}{2}} \left[1 + \frac{1}{\nu}(y_i - \mu)^{\mathrm{T}}\Omega(y_i - \mu)\right]^{-\frac{\nu+k}{2}}, \quad y_i \in \mathbb{R}^k.$$

Consider the random variables $W_i \sim \mathcal{N}_k\left(0, \Omega^{-1}\right)$ and $V_i \sim \chi^2_\nu \equiv \text{Gamma}\left(\frac{\nu}{2}, \frac{1}{2}\right)$. Then, we observe that:

$$Y_i \overset{d}{=} \frac{W_i}{\sqrt{\frac{V_i}{\nu}}} + \mu.$$

We let $Z_i = \frac{V_i}{\nu}$. Then, $Z_i \sim \text{Gamma}\left(\frac{\nu}{2}, \frac{\nu}{2}\right)$. We observe that:

$$Y_i \mid z_i \overset{d}{=} \frac{W_i}{\sqrt{z_i}} + \mu \sim \mathcal{N}_k\left(\mu, z_i^{-1}\Omega^{-1}\right).$$

Suppose that the degrees of freedom ν are known and that the parameters μ, Ω are a priori independent with prior distributions $\mu \sim \mathcal{N}_k\left(a, C^{-1}\right)$ and $\Omega \sim \mathcal{W}_k\left(A^{-1}, d\right)$. Calculate the conditional posterior distributions of the parameters μ, Ω and the latent variables z_i.

Solution.

The joint prior distribution may be written as follows:

$$\begin{aligned}
\pi(\mu, \Omega) &= \pi(\mu) \cdot \pi(\Omega) \\
&= (2\pi)^{-\frac{k}{2}} \left|C^{-1}\right|^{-\frac{1}{2}} \exp\left\{-\frac{(\mu - a)^{\mathrm{T}}C(\mu - a)}{2}\right\} \cdot \frac{|A|^{\frac{d}{2}}}{2^{\frac{dk}{2}}\Gamma_k\left(\frac{d}{2}\right)} |\Omega|^{\frac{d-k-1}{2}} e^{-\frac{1}{2}\mathrm{tr}(A\Omega)} \\
&\propto \exp\left\{-\frac{\mu^{\mathrm{T}}C\mu - 2\mu^{\mathrm{T}}Ca + a^{\mathrm{T}}Ca}{2}\right\} \cdot |\Omega|^{\frac{d-k-1}{2}} e^{-\frac{1}{2}\mathrm{tr}(A\Omega)} \\
&\propto \exp\left\{-\frac{\mu^{\mathrm{T}}C\mu}{2} + \mu^{\mathrm{T}}Ca\right\} \cdot |\Omega|^{\frac{d-k-1}{2}} e^{-\frac{1}{2}\mathrm{tr}(A\Omega)}.
\end{aligned}$$

We define:

$$\overline{zy} = \frac{1}{n}\sum_{i=1}^{n} z_i y_i.$$

The complete-data likelihood, i.e. the joint likelihood of the observed variables y_i and the latent variables z_i, is given by:

$$\begin{aligned}
f(y, z \mid \mu, \Omega) &= \prod_{i=1}^{n} f(y_i, z_i \mid \mu, \Omega) \\
&= \prod_{i=1}^{n} f(z_i) f(y_i \mid z_i, \mu, \Omega) \\
&= \prod_{i=1}^{n} \frac{\left(\frac{\nu}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} z_i^{\frac{\nu}{2}-1} e^{-\frac{\nu}{2}z_i} (2\pi)^{-\frac{k}{2}} |z_i^{-1}\Omega^{-1}|^{-\frac{1}{2}} \exp\left\{-\frac{z_i(y_i - \mu)^{\mathrm{T}}\Omega(y_i - \mu)}{2}\right\}
\end{aligned}$$

$$\begin{aligned}
&\propto |\Omega|^{\frac{n}{2}} \cdot \prod_{i=1}^{n} z_i^{\frac{\nu+k}{2}-1} \exp\left\{-\sum_{i=1}^{n} \frac{\nu + (y_i-\mu)^{\mathrm{T}}\Omega(y_i-\mu)}{2} z_i\right\} \\
&= |\Omega|^{\frac{n}{2}} \exp\left\{-\frac{1}{2}\mathrm{tr}\left[\sum_{i=1}^{n} z_i(y_i-\mu)(y_i-\mu)^{\mathrm{T}}\Omega\right]\right\} \cdot \prod_{i=1}^{n} z_i^{\frac{\nu+k}{2}-1} \exp\left\{-\frac{\nu}{2}\sum_{i=1}^{n} z_i\right\} \\
&= |\Omega|^{\frac{n}{2}} \exp\left\{-\sum_{i=1}^{n} \frac{y_i^{\mathrm{T}}\Omega y_i - 2\mu^{\mathrm{T}}\Omega y_i + \mu^{\mathrm{T}}\Omega\mu}{2} z_i\right\} \cdot \prod_{i=1}^{n} z_i^{\frac{\nu+k}{2}-1} \exp\left\{-\frac{\nu}{2}\sum_{i=1}^{n} z_i\right\} \\
&= \exp\left\{-\frac{\mu^{\mathrm{T}} n\overline{z}\Omega\mu}{2} + \mu^{\mathrm{T}} n\Omega\overline{zy}\right\} \cdot |\Omega|^{\frac{n}{2}} \exp\left\{-\frac{1}{2}\mathrm{tr}\left(\sum_{i=1}^{n} z_i y_i y_i^{\mathrm{T}}\Omega\right)\right\} \\
&\quad \times \prod_{i=1}^{n} z_i^{\frac{\nu+k}{2}-1} \exp\left\{-\frac{\nu}{2}\sum_{i=1}^{n} z_i\right\}.
\end{aligned}$$

Therefore, we get the conditional posterior distributions of μ and Ω as follows:

$$\begin{aligned}
\pi(\mu \mid \Omega, z, y) &\propto \pi(\mu, \Omega, z \mid y) \\
&\propto \pi(\mu, \Omega) \cdot f(y, z \mid \mu, \Omega) \\
&\propto \exp\left\{-\frac{\mu^{\mathrm{T}} C\mu}{2} + \mu^{\mathrm{T}} Ca\right\} \cdot \exp\left\{-\frac{\mu^{\mathrm{T}} n\overline{z}\Omega\mu}{2} + \mu^{\mathrm{T}} n\Omega\overline{zy}\right\} \\
&= \exp\left\{-\frac{1}{2}\mu^{\mathrm{T}} \underbrace{(C + n\overline{z}\Omega)}_{C_n} \mu + \mu^{\mathrm{T}} (Ca + n\Omega\overline{zy})\right\} \\
&= \exp\left\{-\frac{\mu^{\mathrm{T}} C_n\mu}{2} + \mu^{\mathrm{T}} \underbrace{(C + n\overline{z}\Omega)}_{C_n} \underbrace{(C + n\overline{z}\Omega)^{-1} (Ca + n\Omega\overline{zy})}_{a_n}\right\},
\end{aligned}$$

$$\begin{aligned}
\pi(\Omega \mid \mu, z, y) &\propto \pi(\mu, \Omega) \cdot f(y, z \mid \mu, \Omega) \\
&\propto |\Omega|^{\frac{d-k-1}{2}} e^{-\frac{1}{2}\mathrm{tr}(A\Omega)} \cdot |\Omega|^{\frac{n}{2}} \exp\left\{-\frac{1}{2}\mathrm{tr}\left[\sum_{i=1}^{n} z_i(y_i-\mu)(y_i-\mu)^{\mathrm{T}}\Omega\right]\right\} \\
&= |\Omega|^{\frac{d+n-k-1}{2}} \exp\left\{-\frac{1}{2}\mathrm{tr}\left[\left(A + \sum_{i=1}^{n} z_i(y_i-\mu)(y_i-\mu)^{\mathrm{T}}\right)\Omega\right]\right\}.
\end{aligned}$$

Furthermore, we get the conditional posterior distribution of the latent variables z_i as follows:

$$f(z_i \mid y_i, \mu, \Omega) \propto f(y_i, z_i \mid \mu, \Omega) \propto z_i^{\frac{\nu+k}{2}-1} \exp\left\{-\frac{\nu + (y_i-\mu)^{\mathrm{T}}\Omega(y_i-\mu)}{2} z_i\right\}.$$

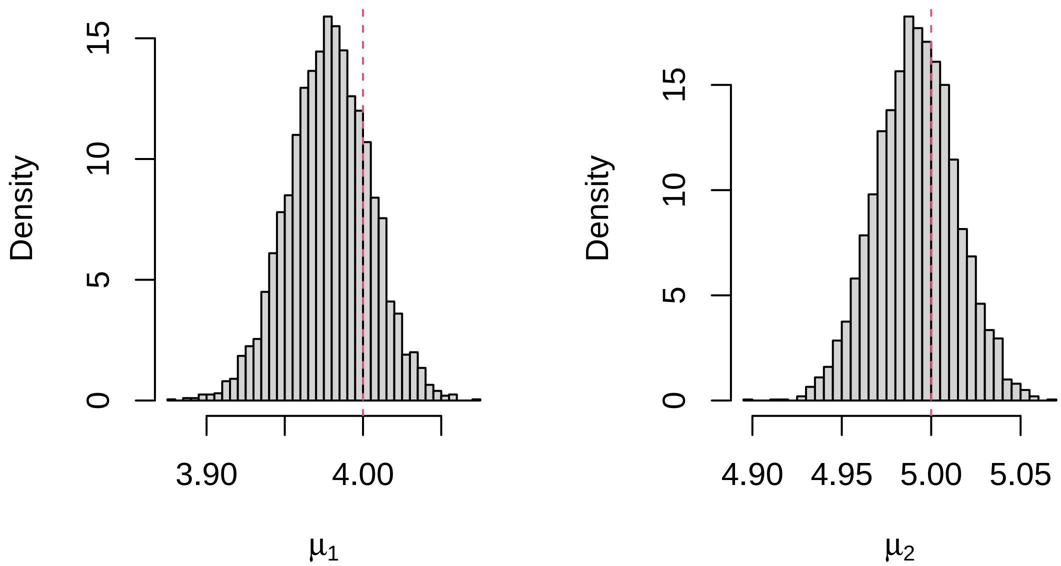
In other words,

$$\mu \mid \Omega, z, y \sim \mathcal{N}_k\left((C + n\overline{z}\Omega)^{-1} (Ca + n\Omega\overline{zy}), (C + n\overline{z}\Omega)^{-1}\right),$$

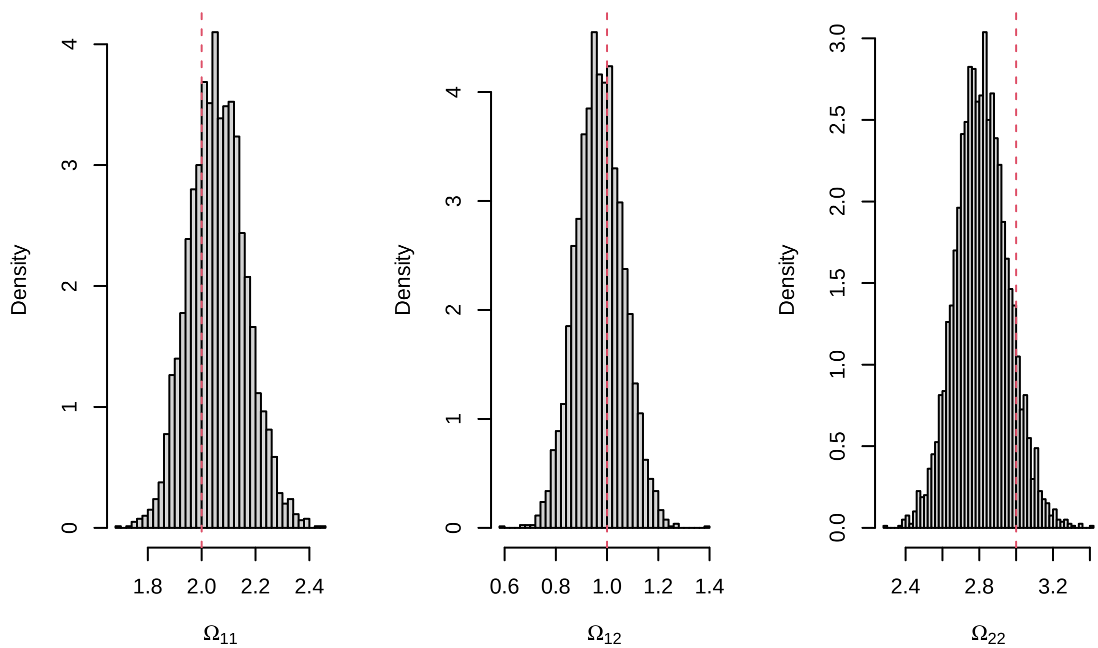
$$\Omega \mid \mu, z, y \sim \mathcal{W}_k\left(\left(A + \sum_{i=1}^{n} z_i(y_i-\mu)(y_i-\mu)^{\mathrm{T}}\right)^{-1}, d+n\right),$$

$$z_i \mid y_i, \mu, \Omega \sim \mathrm{Gamma}\left(\frac{\nu+k}{2}, \frac{\nu + (y_i-\mu)^{\mathrm{T}}\Omega(y_i-\mu)}{2}\right).$$

```r
MCMCmvt = function(Y, mu0, Omega0, nu, a, C, A, d, niter, nburn) {
    library(MASS)
    n = dim(Y)[1]
    k = dim(Y)[2]
    mu = matrix(0, niter, k)
    Omega = array(0, c(k, k, niter))
    Z = matrix(0, niter, n)
    mu[1, ] = mu0
    Omega[, , 1] = Omega0
    Z[1, ] = rgamma(n, (nu + k)/2, (nu + colSums((t(Y) - mu[1, ]) * Omega[,
        , 1] %*% (t(Y) - mu[1, ])))/2)
    for (i in 2:niter) {
        Cn = C + sum(Z[i - 1, ]) * Omega[, , i - 1]
        an = solve(Cn, C %*% a + Omega[, , i - 1] %*% colSums(Z[i - 1, ] * Y))
        mu[i, ] = mvrnorm(1, an, solve(Cn))
        Omega[, , i] = rWishart(1, d + n, solve(A + crossprod(sqrt(Z[i - 1,
            ]) * t(t(Y) - mu[i, ]))))
        Z[i, ] = rgamma(n, (nu + k)/2, (nu + colSums((t(Y) - mu[i, ]) * Omega[,
            , i] %*% (t(Y) - mu[i, ])))/2)
    }
    return(list(mu = mu[-(1:nburn), ], Omega = Omega[, , -(1:nburn)]))
}

library(mvtnorm)
n = 1000
k = 2
mu = c(4, 5)
Omega = matrix(c(2, 1, 1, 3), k)
nu = 10
Y = rmvt(n, solve(Omega), nu, mu)
posterior = MCMCmvt(Y, numeric(k), diag(k), nu, numeric(k), matrix(0, k, k),
    matrix(0, k, k), 2 - k, 5000, 1000)
par(mfrow = c(1, 2))
hist(posterior$mu[, 1], "FD", freq = FALSE, main = NA, xlab = expression(mu[1]))
abline(v = mu[1], col = 2, lty = 2)
hist(posterior$mu[, 2], "FD", freq = FALSE, main = NA, xlab = expression(mu[2]))
abline(v = mu[2], col = 2, lty = 2)
```




```
par(mfrow = c(1, 3))
hist(posterior$Omega[1, 1, ], "FD", freq = FALSE, main = NA, xlab = expression(Omega[11]))
abline(v = Omega[1, 1], col = 2, lty = 2)
hist(posterior$Omega[1, 2, ], "FD", freq = FALSE, main = NA, xlab = expression(Omega[12]))
abline(v = Omega[1, 2], col = 2, lty = 2)
hist(posterior$Omega[2, 2, ], "FD", freq = FALSE, main = NA, xlab = expression(Omega[22]))
abline(v = Omega[2, 2], col = 2, lty = 2)
```

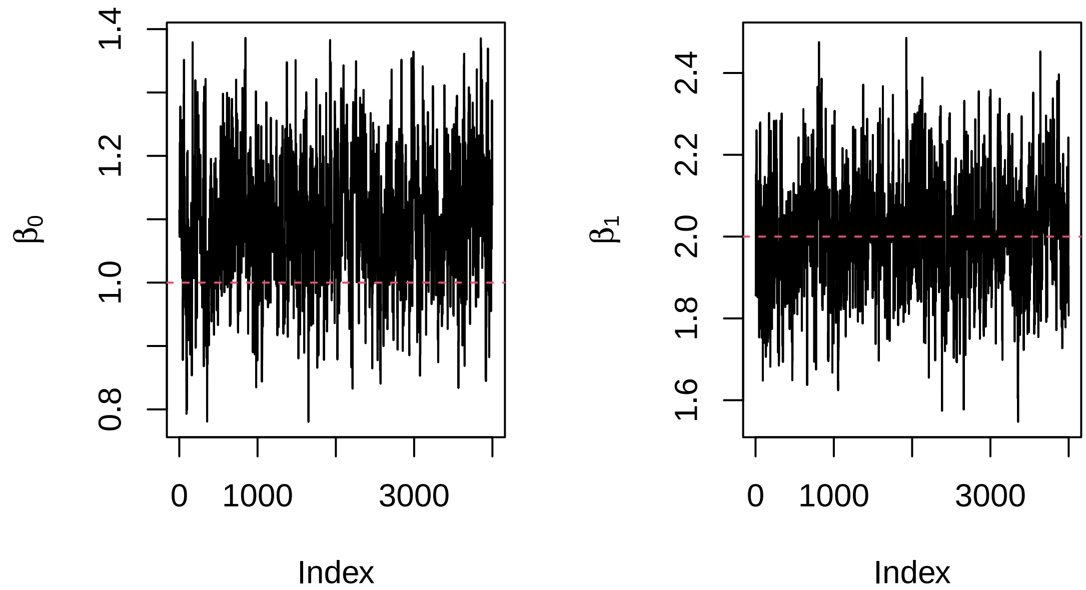


2 Generalized Linear Models

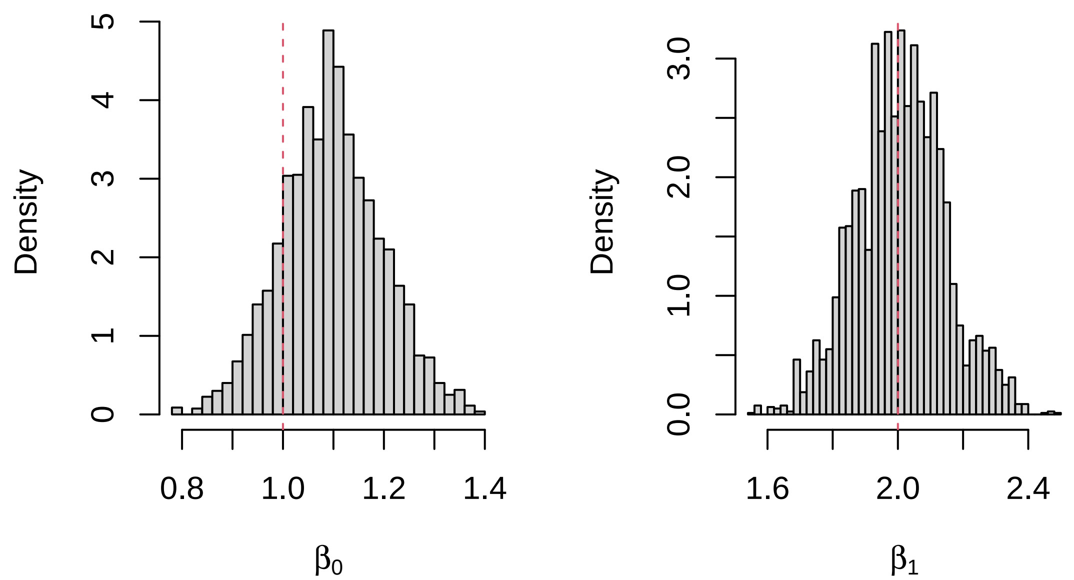
2.1 Logistic Model

Consider the logistic regression model $y_i \sim \text{Bernoulli}(p_i)$, where $\text{logit}\, p_i = \log \frac{p_i}{1-p_i} = \beta_0 + \beta_1 x_i$, i.e. $p_i = \frac{1}{1+e^{-\beta_0 - \beta_1 x_i}}$. We consider prior independence with improper prior distributions $\pi(\beta_0) \propto 1$ and $\pi(\beta_1) \propto 1$. We can implement a Random Walk Metropolis-Hastings Algorithm with proposed random variables $\beta_0^\star \mid \beta_0^{(\ell-1)} \sim \mathcal{N}\left(\beta_0^{(\ell-1)}, \sigma_0^2\right)$ and $\beta_1^\star \mid \beta_1^{(\ell-1)} \sim \mathcal{N}\left(\beta_1^{(\ell-1)}, \sigma_1^2\right)$.

```
RWMHlogistic = function(Y, X, beta00, beta10, beta0sd, beta1sd, niter, nburn) {
    beta0 = numeric(niter)
    beta1 = numeric(niter)
    beta0[1] = beta00
    beta1[1] = beta10
    for (i in 2:niter) {
        beta0star = rnorm(1, beta0[i - 1], beta0sd)
        logA = sum(dbinom(Y, 1, (1 + exp(-beta0star - beta1[i - 1] * X))^(-1),
            log = TRUE) - dbinom(Y, 1, (1 + exp(-beta0[i - 1] - beta1[i - 1] *
            X))^(-1), log = TRUE))
        beta0[i] = ifelse(log(runif(1)) < logA, beta0star, beta0[i - 1])
        beta1star = rnorm(1, beta1[i - 1], beta1sd)
        logA = sum(dbinom(Y, 1, (1 + exp(-beta0[i] - beta1star * X))^(-1), log = TRUE) -
            dbinom(Y, 1, (1 + exp(-beta0[i] - beta1[i - 1] * X))^(-1), log = TRUE))
        beta1[i] = ifelse(log(runif(1)) < logA, beta1star, beta1[i - 1])
    }
    return(list(beta0 = beta0[-(1:nburn)], beta1 = beta1[-(1:nburn)]))
}

n = 1000
beta0 = 1
beta1 = 2
X = rnorm(n)
Y = rbinom(n, 1, (1 + exp(-beta0 - beta1 * X))^(-1))
posterior = RWMHlogistic(Y, X, 0, 0, 0.15, 0.25, 5000, 1000)
par(mfrow = c(1, 2))
plot(posterior$beta0, type = "l", ylab = expression(beta[0]))
abline(h = beta0, col = 2, lty = 2)
plot(posterior$beta1, type = "l", ylab = expression(beta[1]))
abline(h = beta1, col = 2, lty = 2)
```




```
hist(posterior$beta0, "FD", freq = FALSE, main = NA, xlab = expression(beta[0]))
abline(v = beta0, col = 2, lty = 2)
hist(posterior$beta1, "FD", freq = FALSE, main = NA, xlab = expression(beta[1]))
abline(v = beta1, col = 2, lty = 2)
```



2.2 Probit Model

Consider the probit regression model $y_i \sim \text{Bernoulli}(p_i)$, where $p_i = \Phi\left(x_i^{\mathrm{T}}\beta\right)$ and $\beta \in \mathbb{R}^k$. We consider the independent random variables $z_i = x_i^{\mathrm{T}}\beta + \varepsilon_i$, where $\varepsilon_i \sim \mathcal{N}(0,1)$. Then, we observe that:

$$p_i = \mathbb{P}(Y_i = 1) = \Phi\left(x_i^{\mathrm{T}}\beta\right) = \mathbb{P}(\varepsilon_i \leqslant x_i^{\mathrm{T}}\beta) = \mathbb{P}\left(\varepsilon_i \geqslant -x_i^{\mathrm{T}}\beta\right) = \mathbb{P}(Z_i \geqslant 0).$$

In other words, $y_i \mid z_i \overset{d}{=} \mathbb{1}_{\{z_i \geqslant 0\}}$. We consider the prior distribution $\beta \sim \mathcal{N}_k\left(a, C^{-1}\right)$. Calculate the conditional posterior distributions of the parameter β and the latent variables z_i.

Solution.

The prior distribution may be written as follows:

$$\begin{aligned}
\pi(\beta) &= (2\pi)^{-\frac{k}{2}} \left|C^{-1}\right|^{-\frac{1}{2}} \exp\left\{-\frac{(\beta-a)^{\mathrm{T}}C(\beta-a)}{2}\right\} \\
&\propto \exp\left\{-\frac{\beta^{\mathrm{T}}C\beta - 2\beta^{\mathrm{T}}Ca + a^{\mathrm{T}}Ca}{2}\right\} \\
&\propto \exp\left\{-\frac{\beta^{\mathrm{T}}C\beta}{2} + \beta^{\mathrm{T}}Ca\right\}.
\end{aligned}$$

The complete-data likelihood, i.e. the joint likelihood of the observed variables y_i and the latent variables z_i, is given by:

$$\begin{aligned}
f(y, z \mid \beta) &= \prod_{i=1}^{n} f(y_i, z_i \mid \beta) \\
&= \prod_{i=1}^{n} f(z_i \mid \beta) f(y_i \mid z_i) \\
&= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\left(z_i - x_i^{\mathrm{T}}\beta\right)^2}{2}\right\} \mathbb{1}_{\{z_i \geqslant 0\}}^{y_i} \mathbb{1}_{\{z_i < 0\}}^{1-y_i}.
\end{aligned}$$

We define $z = (z_1, \ldots, z_n)^{\mathrm{T}} \in \mathbb{R}^n$ and the design matrix $X = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^{n \times k}$. Then, we observe that $z \sim N_n(X\beta, I_n)$. In other words, the likelihood of the sample is given by:

$$\begin{aligned}
f(y, z \mid \beta) &= f(z \mid \beta) \cdot f(y \mid z) \\
&= f(z \mid \beta) \cdot \prod_{i=1}^{n} f(y_i \mid z_i) \\
&= (2\pi)^{-\frac{n}{2}} \left|I_n\right|^{-\frac{1}{2}} \exp\left\{-\frac{(z - X\beta)^{\mathrm{T}}(z - X\beta)}{2}\right\} \cdot \prod_{i=1}^{n} \mathbb{1}_{\{z_i \geqslant 0\}}^{y_i} \mathbb{1}_{\{z_i < 0\}}^{1-y_i} \\
&\propto \exp\left\{-\frac{z^{\mathrm{T}}z - 2\beta^{\mathrm{T}}X^{\mathrm{T}}z + \beta^{\mathrm{T}}X^{\mathrm{T}}X\beta}{2}\right\} \cdot \prod_{i=1}^{n} \mathbb{1}_{\{z_i \geqslant 0\}}^{y_i} \mathbb{1}_{\{z_i < 0\}}^{1-y_i} \\
&\propto \exp\left\{-\frac{\beta^{\mathrm{T}}X^{\mathrm{T}}X\beta}{2} + \beta^{\mathrm{T}}X^{\mathrm{T}}z\right\} \cdot \prod_{i=1}^{n} \mathbb{1}_{\{z_i \geqslant 0\}}^{y_i} \mathbb{1}_{\{z_i < 0\}}^{1-y_i}.
\end{aligned}$$

Therefore, we get the conditional posterior distribution of β as follows:

$$\begin{aligned}
\pi(\beta \mid z, y) &\propto \pi(\beta, z \mid y) \\
&\propto \pi(\beta) \cdot f(y, z \mid \beta) \\
&\propto \exp\left\{-\frac{\beta^{\mathrm{T}}C\beta}{2} + \beta^{\mathrm{T}}Ca\right\} \cdot \exp\left\{-\frac{\beta^{\mathrm{T}}X^{\mathrm{T}}X\beta}{2} + \beta^{\mathrm{T}}X^{\mathrm{T}}z\right\} \\
&= \exp\left\{-\frac{1}{2}\beta^{\mathrm{T}} \underbrace{\left(C + X^{\mathrm{T}}X\right)}_{C_n} \beta + \beta^{\mathrm{T}}\left(Ca + X^{\mathrm{T}}z\right)\right\} \\
&= \exp\left\{-\frac{\beta^{\mathrm{T}}C_n\beta}{2} + \beta^{\mathrm{T}} \underbrace{\left(C + X^{\mathrm{T}}X\right)}_{C_n} \underbrace{\left(C + X^{\mathrm{T}}X\right)^{-1}\left(Ca + X^{\mathrm{T}}z\right)}_{a_n}\right\}.
\end{aligned}$$

Furthermore, we get the conditional posterior distribution of the latent variables z_i as follows:

$$f(z_i \mid y_i, \beta) \propto f(y_i, z_i \mid \beta) \propto \exp\left\{-\frac{\left(z_i - x_i^{\mathrm{T}}\beta\right)^2}{2}\right\} \mathbb{1}_{\{z_i \geqslant 0\}}^{y_i} \mathbb{1}_{\{z_i < 0\}}^{1-y_i}.$$

In other words,

$$\beta \mid z \sim \mathcal{N}_k\left(\left(C + X^{\mathrm{T}}X\right)^{-1}\left(Ca + X^{\mathrm{T}}z\right), \left(C + X^{\mathrm{T}}X\right)^{-1}\right),$$

$$(z_i \mid y_i = 1, \beta) \sim \mathcal{N}\left(x_i^{\mathrm{T}}\beta, 1\right) \mathbb{1}_{\{z_i \geqslant 0\}}, \quad (z_i \mid y_i = 0, \beta) \sim \mathcal{N}\left(x_i^{\mathrm{T}}\beta, 1\right) \mathbb{1}_{\{z_i < 0\}}.$$

We observe that:

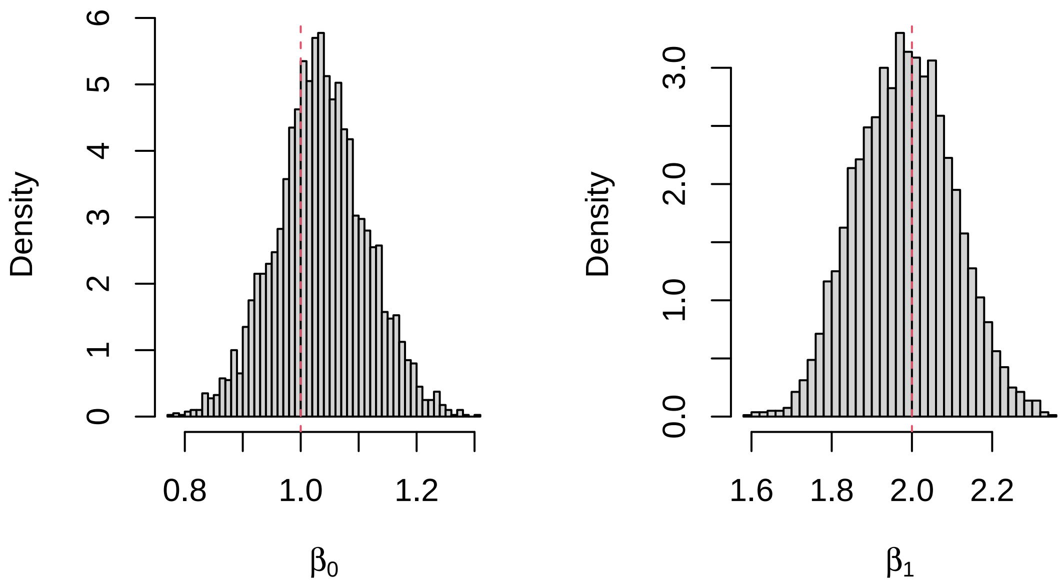
$$F_{Z_i \mid y_i = 1, \beta}(z_i) = \frac{\Phi\left(z_i - x_i^{\mathrm{T}}\beta\right) - \Phi\left(-x_i^{\mathrm{T}}\beta\right)}{1 - \Phi\left(-x_i^{\mathrm{T}}\beta\right)} \mathbb{1}_{\{z_i \geqslant 0\}}, \quad F_{Z_i \mid y_i = 0, \beta}(z_i) = \frac{\Phi\left(z_i - x_i^{\mathrm{T}}\beta\right)}{\Phi\left(-x_i^{\mathrm{T}}\beta\right)} \mathbb{1}_{\{z_i < 0\}}.$$

If $U_1, U_2, \ldots, U_n \sim \mathrm{Unif}[0, 1]$, then we get that:

$$Z_i = \begin{cases} \Phi^{-1}\left[\Phi\left(x_i^{\mathrm{T}}\beta\right) U_i + 1 - \Phi\left(x_i^{\mathrm{T}}\beta\right)\right] + x_i^{\mathrm{T}}\beta, & y_i = 1 \\ \Phi^{-1}\left[\left(1 - \Phi\left(x_i^{\mathrm{T}}\beta\right)\right) U_i\right] + x_i^{\mathrm{T}}\beta, & y_i = 0 \end{cases}.$$

```
MCMCprobit = function(Y, X, beta0, a, C, niter, nburn) {
    library(MASS)
    n = length(Y)
    k = dim(X)[2]
    beta = matrix(0, niter, k)
    Z = matrix(0, niter, n)
    beta[1, ] = beta0
    prob = pnorm(X %*% beta[1, ])
    U = runif(n)
    Z[1, ] = X %*% beta[1, ] + ifelse(Y == 1, qnorm(prob * U + 1 - prob), qnorm((1 -
        prob) * U))
    for (i in 2:niter) {
        Cninv = solve(C + crossprod(X))
        an = crossprod(Cninv, C %*% a + crossprod(X, Z[i - 1, ]))
        beta[i, ] = mvrnorm(1, an, Cninv)
        prob = pnorm(X %*% beta[i, ])
        U = runif(n)
        Z[i, ] = X %*% beta[i, ] + ifelse(Y == 1, qnorm(prob * U + 1 - prob),
            qnorm((1 - prob) * U))
    }
    return(list(beta = beta[-(1:nburn), ], Z = Z[-(1:nburn), ]))
}

n = 1000
k = 2
beta = c(1, 2)
```

```
X = cbind(1, rnorm(n))
Y = rbinom(n, 1, pnorm(X %*% beta))
posterior = MCMCprobit(Y, X, numeric(k), numeric(k), matrix(0, k, k), 5000,
    1000)
par(mfrow = c(1, 2))
hist(posterior$beta[, 1], "FD", freq = FALSE, main = NA, xlab = expression(beta[0]))
abline(v = beta[1], col = 2, lty = 2)
hist(posterior$beta[, 2], "FD", freq = FALSE, main = NA, xlab = expression(beta[1]))
abline(v = beta[2], col = 2, lty = 2)
```

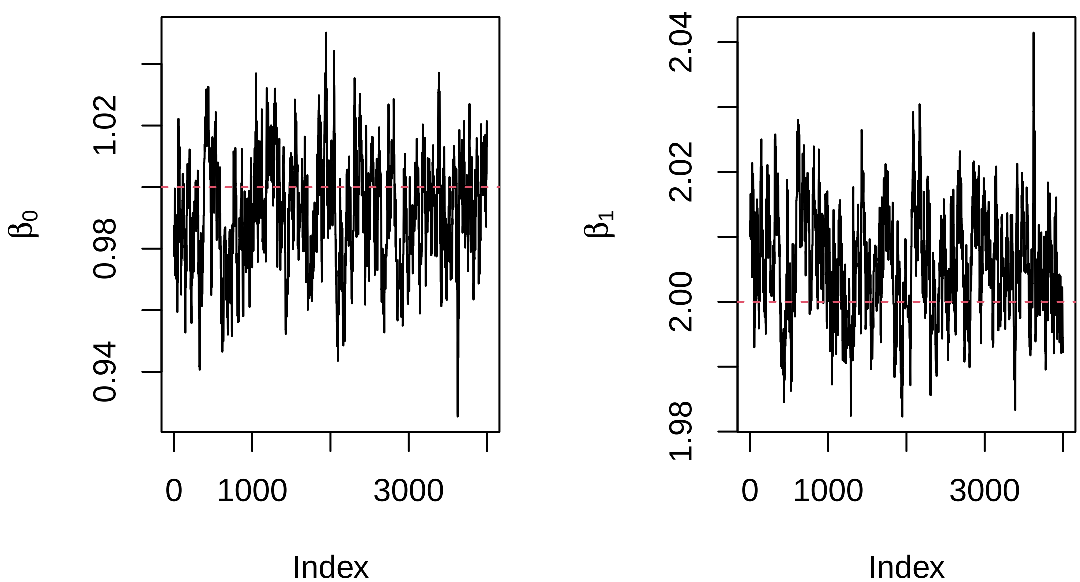


2.3 Log-Linear Poisson Model

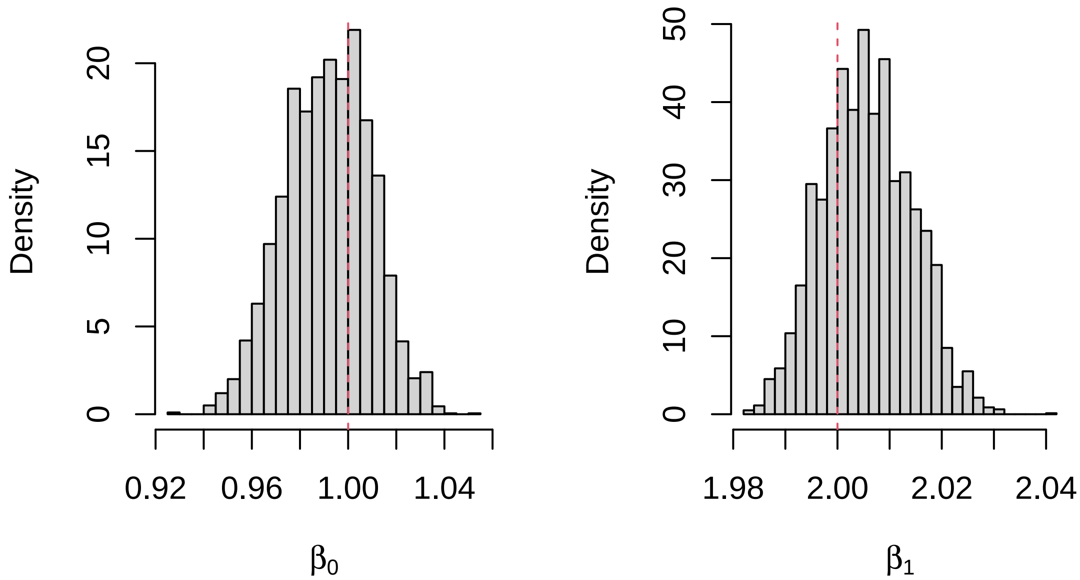
Consider the log-linear regression model $y_i \sim \text{Poisson}(\lambda_i)$, where $\log \lambda_i = \beta_0 + \beta_1 x_i$, i.e. $\lambda_i = e^{\beta_0 + \beta_1 x_i}$. We consider prior independence with improper prior distributions $\pi(\beta_0) \propto 1$ and $\pi(\beta_1) \propto 1$. We can implement a Random Walk Metropolis-Hastings algorithm with proposed random variables $\beta_0^\star \mid \beta_0^{(\ell-1)} \sim \mathcal{N}\left(\beta_0^{(\ell-1)}, \sigma_0^2\right)$ and $\beta_1^\star \mid \beta_1^{(\ell-1)} \sim \mathcal{N}\left(\beta_1^{(\ell-1)}, \sigma_1^2\right)$.

```
RWMHpois = function(Y, X, beta00, beta10, beta0sd, beta1sd, niter, nburn) {
    beta0 = numeric(niter)
    beta1 = numeric(niter)
    beta0[1] = beta00
    beta1[1] = beta10
    for (i in 2:niter) {
        beta0star = rnorm(1, beta0[i - 1], beta0sd)
        logA = sum(dpois(Y, exp(beta0star + beta1[i - 1] * X), log = TRUE) -
            dpois(Y, exp(beta0[i - 1] + beta1[i - 1] * X), log = TRUE))
        beta0[i] = ifelse(log(runif(1)) < logA, beta0star, beta0[i - 1])
        beta1star = rnorm(1, beta1[i - 1], beta1sd)
        logA = sum(dpois(Y, exp(beta0[i] + beta1star * X), log = TRUE) - dpois(Y,
```

```
            exp(beta0[i] + beta1[i - 1] * X), log = TRUE))
        beta1[i] = ifelse(log(runif(1)) < logA, beta1star, beta1[i - 1])
    }
    return(list(beta0 = beta0[-(1:nburn)], beta1 = beta1[-(1:nburn)]))
}

n = 1000
beta0 = 1
beta1 = 2
X = rnorm(n)
Y = rpois(n, exp(beta0 + beta1 * X))
posterior = RWMHpois(Y, X, 0, 0, 0.015, 0.0075, 5000, 1000)
par(mfrow = c(1, 2))
plot(posterior$beta0, type = "l", ylab = expression(beta[0]))
abline(h = beta0, col = 2, lty = 2)
plot(posterior$beta1, type = "l", ylab = expression(beta[1]))
abline(h = beta1, col = 2, lty = 2)
```




```
hist(posterior$beta0, "FD", freq = FALSE, main = NA, xlab = expression(beta[0]))
abline(v = beta0, col = 2, lty = 2)
hist(posterior$beta1, "FD", freq = FALSE, main = NA, xlab = expression(beta[1]))
abline(v = beta1, col = 2, lty = 2)
```



2.4 Zero-Inflated Poisson Model

Consider the zero-inflated Poisson regression model:

$$\mathbb{P}(Y_i = k) = \begin{cases} 1 - p + pe^{-\lambda}, & k = 0 \\ pe^{-\lambda}\frac{\lambda^k}{k!}, & k = 1, 2, \dots \end{cases}$$

We consider the independent random variables $z_i \sim \text{Bernoulli}(p)$. For $k = 0, 1, \dots$, we observe that:

$$\mathbb{P}\left(Y_i = k \mid Z_i = 1\right) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad \mathbb{P}\left(Y_i = 0 \mid Z_i = 0\right) = 1.$$

In other words, it holds that $(y_i \mid z_i = 1) \sim \text{Poisson}(\lambda)$ and $(y_i \mid z_i = 0) \overset{d}{=} 0$. We consider prior independence with prior distributions $p \sim \text{Beta}(a, c)$ and $\lambda \sim \text{Gamma}(d, q)$. Calculate the conditional posterior distributions of the parameters p, λ and the latent variables z_i.

Solution.

The joint prior distribution may be written as follows:

$$\begin{aligned} \pi(p, \lambda) &= \pi(p)\pi(\lambda) \\ &= \frac{\Gamma(a+c)}{\Gamma(a)\Gamma(c)} p^{a-1}(1-p)^{c-1} \cdot \frac{q^d}{\Gamma(d)}\lambda^{d-1}e^{-q\lambda} \\ &\propto p^{a-1}(1-p)^{c-1} \cdot \lambda^{d-1}e^{-q\lambda}. \end{aligned}$$

The complete-data likelihood, i.e. the joint likelihood of the observed variables y_i and the latent variables z_i, is given by:

$$f(y, z \mid p, \lambda) = \prod_{i=1}^{n} f(y_i, z_i \mid p, \lambda)$$

$$
\begin{aligned}
&= \prod_{i=1}^{n} f(z_i \mid p) f(y_i \mid z_i, \lambda) \\
&= \prod_{i=1}^{n} p^{z_i}(1-p)^{1-z_i} \left(e^{-\lambda}\frac{\lambda^{y_i}}{y_i!}\right)^{z_i} \mathbb{1}_{\{y_i=0\}}^{1-z_i} \\
&\propto \prod_{i=1}^{n} p^{z_i}(1-p)^{1-z_i} e^{-\lambda z_i}\lambda^{y_i}\mathbb{1}_{\{y_i=0\}}^{1-z_i} \\
&\propto p^{n\overline{z}}(1-p)^{n-n\overline{z}} \cdot \lambda^{n\overline{y}} e^{-n\lambda\overline{z}} \cdot \prod_{i=1}^{n} \mathbb{1}_{\{y_i=0\}}^{1-z_i}.
\end{aligned}
$$

Therefore, we get the conditional posterior distributions of p and λ as follows:

$$
\begin{aligned}
\pi(p \mid \lambda, z, y) &\propto \pi(p, \lambda, z \mid y) \\
&\propto \pi(p) \cdot f(y, z \mid p, \lambda) \\
&\propto p^{a-1}(1-p)^{c-1} \cdot p^{n\overline{z}}(1-p)^{n-n\overline{z}} \\
&= p^{a+n\overline{z}-1}(1-p)^{c+n-n\overline{z}-1},
\end{aligned}
$$

$$
\begin{aligned}
\pi(\lambda \mid p, z, y) &\propto \pi(\lambda) \cdot f(y, z \mid p, \lambda) \\
&\propto \lambda^{d-1}e^{-q\lambda} \cdot \lambda^{n\overline{y}} e^{-n\lambda\overline{z}} \\
&= \lambda^{d+n\overline{y}-1} e^{-(q+n\overline{z})\lambda}.
\end{aligned}
$$

Furthermore, we get the conditional posterior distribution of the latent variables z_i as follows:

$$
f(z_i \mid y_i, p, \lambda) \propto f(y_i, z_i \mid p, \lambda) \propto p^{z_i}(1-p)^{1-z_i} e^{-\lambda z_i} \mathbb{1}_{\{y_i=0\}}^{1-z_i} = \left(pe^{-\lambda}\right)^{z_i}(1-p)^{1-z_i}\mathbb{1}_{\{y_i=0\}}^{1-z_i}.
$$

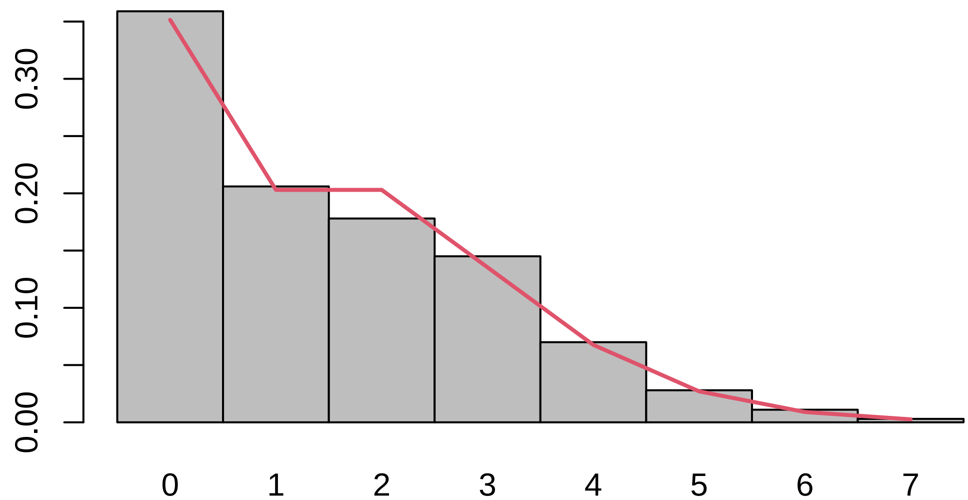
In other words,

$$
p \mid z, y \sim \text{Beta}\left(a + n\overline{z}, c + n - n\overline{z}\right), \quad \lambda \mid z, y \sim \text{Gamma}\left(d + n\overline{y}, q + n\overline{z}\right),
$$

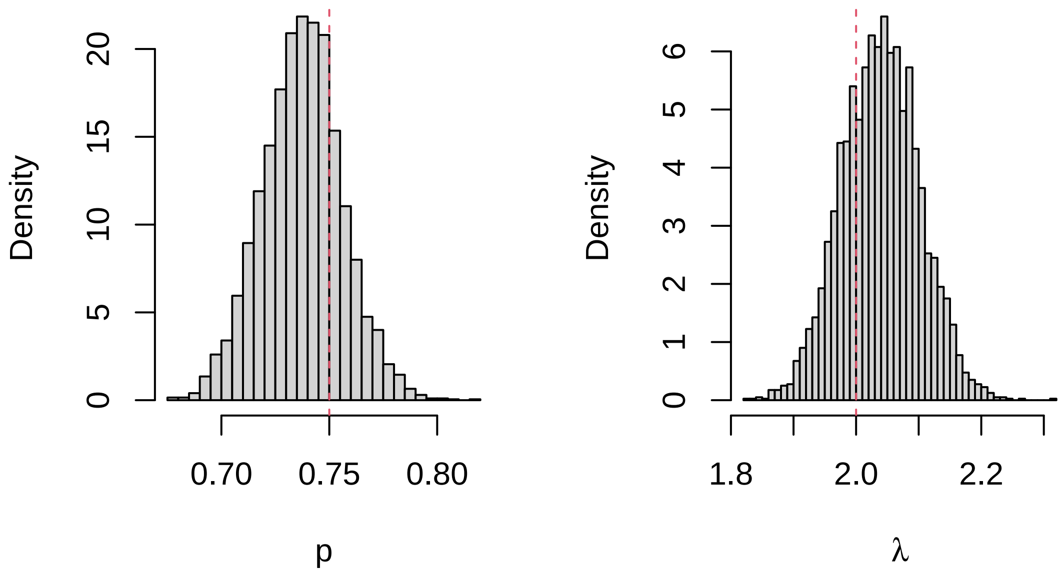
$$
(z_i \mid y_i = 0, p, \lambda) \sim \text{Bernoulli}\left(\frac{pe^{-\lambda}}{pe^{-\lambda} + 1 - p}\right), \quad (z_i \mid y_i > 0, p, \lambda) \overset{d}{=} 1.
$$

```
MCMCzip = function(Y, p0, lambda0, a, c, p, q, niter, nburn) {
    n = length(Y)
    SY = sum(Y)
    p = numeric(niter)
    lambda = numeric(niter)
    Z = matrix(0, niter, n)
    p[1] = p0
    lambda[1] = lambda0
    Z[1, ] = ifelse(Y == 0, rbinom(n, 1, p[1] * exp(-lambda[1])/(p[1] * exp(-lambda[1]) +
        1 - p[1])), 1)
    for (i in 2:niter) {
        SZ = sum(Z[i - 1, ])
        p[i] = rbeta(1, a + SZ, c + n - SZ)
        lambda[i] = rgamma(1, p + SY, q + SZ)
```

```
        Z[i, ] = ifelse(Y == 0, rbinom(n, 1, p[i] * exp(-lambda[i])/(p[i] *
            exp(-lambda[i]) + 1 - p[i])), 1)
    }
    return(list(p = p[-(1:nburn)], lambda = lambda[-(1:nburn)], Z = Z[-(1:nburn),
        ]))
}

n = 1000
p = 0.75
lambda = 2
Y = ifelse(rbinom(n, 1, p) == 1, rpois(n, lambda), 0)
barplot(table(factor(Y, levels = 0:max(Y)))/n, space = 0)
lines(0:max(Y) + 0.5, c(1 - p, numeric(max(Y))) + p * dpois(0:max(Y), lambda),
    col = 2, lwd = 2)
```




```
posterior = MCMCzip(Y, 0.5, 1, 0.5, 0.5, 0.5, 0, 5000, 1000)
par(mfrow = c(1, 2))
hist(posterior$p, "FD", freq = FALSE, main = NA, xlab = "p")
abline(v = p, col = 2, lty = 2)
hist(posterior$lambda, "FD", freq = FALSE, main = NA, xlab = expression(lambda))
abline(v = lambda, col = 2, lty = 2)
```


Density
0
5
10
15
20
0.70
0.75
0.80
p
Density
0
1
2
3
4
5
6
1.8
2.0
2.2
λ

3 Other Applications

3.1 Change Point Model

Consider the following model:

- For $i = 1, 2, \ldots, t$, the observation x_i is an independent realization of a Poisson random variable with mean θ_1.
- For $i = t+1, t+2, \ldots, n$, the observation x_i is an independent realization of a Poisson random variable with mean θ_2.

We consider the prior distributions $\theta_1 \sim \text{Gamma}(p_1, q_1)$, $\theta_2 \sim \text{Gamma}(p_2, q_2)$ and $t \sim U\{1, 2, \ldots, n-1\}$. Calculate the conditional posterior distributions of the parameters of the model and the marginal posterior distribution of t.

Solution.

The joint prior distribution of θ_1, θ_2 and t may be written as:

$$\begin{aligned}\pi(\theta_1, \theta_2, t) &= \pi(\theta_1) \cdot \pi(\theta_2) \cdot \pi(t) \\ &= \frac{q_1^{p_1}}{\Gamma(p_1)} \theta_1^{p_1 - 1} e^{-q_1 \theta_1} \cdot \frac{q_2^{p_2}}{\Gamma(p_2)} \theta_2^{p_2 - 1} e^{-q_2 \theta_2} \cdot \frac{1}{n-1} \\ &\propto \theta_1^{p_1 - 1} e^{-q_1 \theta_1} \cdot \theta_2^{p_2 - 1} e^{-q_2 \theta_2}.\end{aligned}$$

We define:

$$S_t = \sum_{i=1}^{t} x_i.$$

Then, we observe that:

$$S_n - S_t = \sum_{i=1}^{n} x_i - \sum_{i=1}^{t} x_i = \sum_{i=t+1}^{n} x_i.$$

The likelihood of the sample is given by:

$$\begin{aligned}f(x \mid \theta_1, \theta_2, t) &= \prod_{i=1}^{t} f(x_i \mid \theta_1) \cdot \prod_{i=t+1}^{n} f(x_i \mid \theta_2) \\ &= \prod_{i=1}^{t} e^{-\theta_1} \frac{\theta_1^{x_i}}{x_i!} \cdot \prod_{i=t+1}^{n} e^{-\theta_2} \frac{\theta_2^{x_i}}{x_i!} \\ &= e^{-t\theta_1} \theta_1^{S_t} \cdot e^{-(n-t)\theta_2} \theta_2^{S_n - S_t} \cdot \prod_{i=1}^{n} \frac{1}{x_i!} \\ &\propto e^{-t\theta_1} \theta_1^{S_t} \cdot e^{-(n-t)\theta_2} \theta_2^{S_n - S_t}.\end{aligned}$$

Therefore, we get the conditional posterior distributions of θ_1, θ_2 and t as follows:

$$\begin{aligned}\pi(\theta_1 \mid \theta_2, t, x) &\propto \pi(\theta_1, \theta_2, t \mid x) \\ &\propto \pi(\theta_1, \theta_2, t) \cdot f(x \mid \theta_1, \theta_2, t) \\ &\propto \theta_1^{p_1 - 1} e^{-q_1 \theta_1} \cdot e^{-t\theta_1} \theta_1^{S_t} \\ &= \theta_1^{p_1 + S_t - 1} e^{-(q_1 + t)\theta_1},\end{aligned}$$

$$
\begin{aligned}
\pi(\theta_2 \mid \theta_1, t, x) &\propto \pi(\theta_1, \theta_2, t) \cdot f(x \mid \theta_1, \theta_2, t) \\
&\propto \theta_2^{p_2-1} e^{-q_2\theta_2} \cdot e^{-(n-t)\theta_2} \theta_2^{S_n - S_t} \\
&= \theta_2^{p_2+S_n-S_t-1} e^{-(q_2+n-t)\theta_2},
\end{aligned}
$$

$$
\begin{aligned}
\pi(t \mid \theta_1, \theta_2, x) &\propto \pi(\theta_1, \theta_2, t) \cdot f(x \mid \theta_1, \theta_2, t) \\
&\propto e^{-t\theta_1} \theta_1^{S_t} \cdot e^{-(n-t)\theta_2} \theta_2^{S_n-S_t} \\
&\propto e^{-t\theta_1} \theta_1^{S_t} \cdot e^{t\theta_2} \theta_2^{-S_t} \\
&= e^{-(\theta_1-\theta_2)t} \left(\frac{\theta_1}{\theta_2}\right)^{S_t}.
\end{aligned}
$$

We observe that the parameters θ_1 and θ_2 are a posteriori independent given t, that is:

$$
\theta_1 \mid t, x \sim \text{Gamma}\left(p_1 + S_t, q_1 + t\right), \quad \theta_2 \mid t, x \sim \text{Gamma}\left(p_2 + S_n - S_t, q_2 + n - t\right).
$$

The conditional posterior distribution of t is a discrete distribution $\pi(t \mid \theta_1, \theta_2, x)$ with finite support $\{1, 2, \dots, n-1\}$. For the calculation of the probability vector $\pi\left(t \mid \theta_1, \theta_2, x\right)$ we use the Log-Sum-Exp trick. In other words, we define:

$$
v_i = -(\theta_1 - \theta_2)i + S_i \log \frac{\theta_1}{\theta_2}, \quad m = \max_{i \in \{1, \dots, n-1\}} v_i.
$$

Then, we get that:

$$
\pi\left(t \mid \theta_1, \theta_2, x\right) = \frac{e^{v_t - m}}{\sum_{i=1}^{n-1} e^{v_i - m}}.
$$

Furthermore, we get the marginal posterior distribution of t as follows:

$$
\begin{aligned}
\pi(t \mid x) &= \int_0^\infty \int_0^\infty \pi(\theta_1, \theta_2, t \mid x) d\theta_1 d\theta_2 \\
&\propto \int_0^\infty \int_0^\infty \pi(\theta_1, \theta_2, t) f(x \mid \theta_1, \theta_2, t) d\theta_1 d\theta_2 \\
&\propto \int_0^\infty \int_0^\infty \theta_1^{p_1-1} e^{-q_1\theta_1} \cdot \theta_2^{p_2-1} e^{-q_2\theta_2} \cdot e^{-t\theta_1} \theta_1^{S_t} \cdot e^{-(n-t)\theta_2} \theta_2^{S_n-S_t} d\theta_1 d\theta_2 \\
&= \int_0^\infty \theta_1^{p_1+S_t-1} e^{-(q_1+t)\theta_1} d\theta_1 \cdot \int_0^\infty \theta_2^{p_2+S_n-S_t-1} e^{-(q_2+n-t)\theta_2} d\theta_2 \\
&= \frac{\Gamma(p_1+S_t)}{(q_1+t)^{p_1+S_t}} \cdot \frac{\Gamma(p_2+S_n-S_t)}{(q_2+n-t)^{p_2+S_n-S_t}}.
\end{aligned}
$$

We implement the following Gibbs sampler to simulate from this joint posterior distribution.

```
MCMCchangepoint = function(Y, theta10, theta20, p1, q1, p2, q2, niter, nburn) {
    n = length(Y)
    S = cumsum(Y)
    theta1 = numeric(niter)
    theta2 = numeric(niter)
    t = numeric(niter)
    theta1[1] = theta10
    theta2[1] = theta20
```

Algorithm 3.1 Gibbs Sampler

Initialize $\theta_1^{(0)}$, $\theta_2^{(0)}$, $t^{(0)}$.

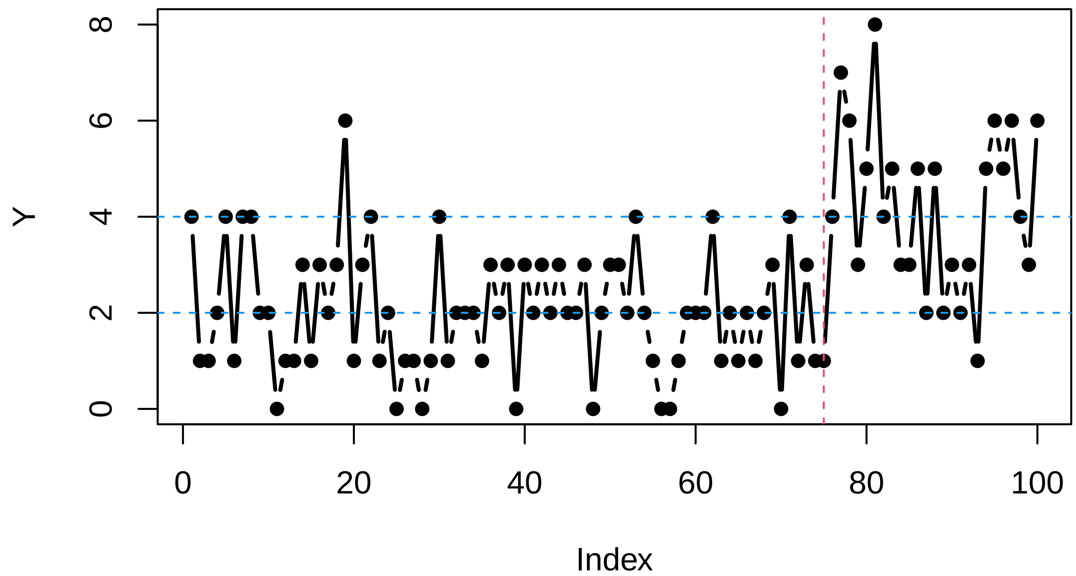
Iterate the following steps:

1: Simulate $\theta_1^{(k)} \sim \text{Gamma}\left(p_1 + S_{t^{(k-1)}}, q_1 + t^{(k-1)}\right)$.

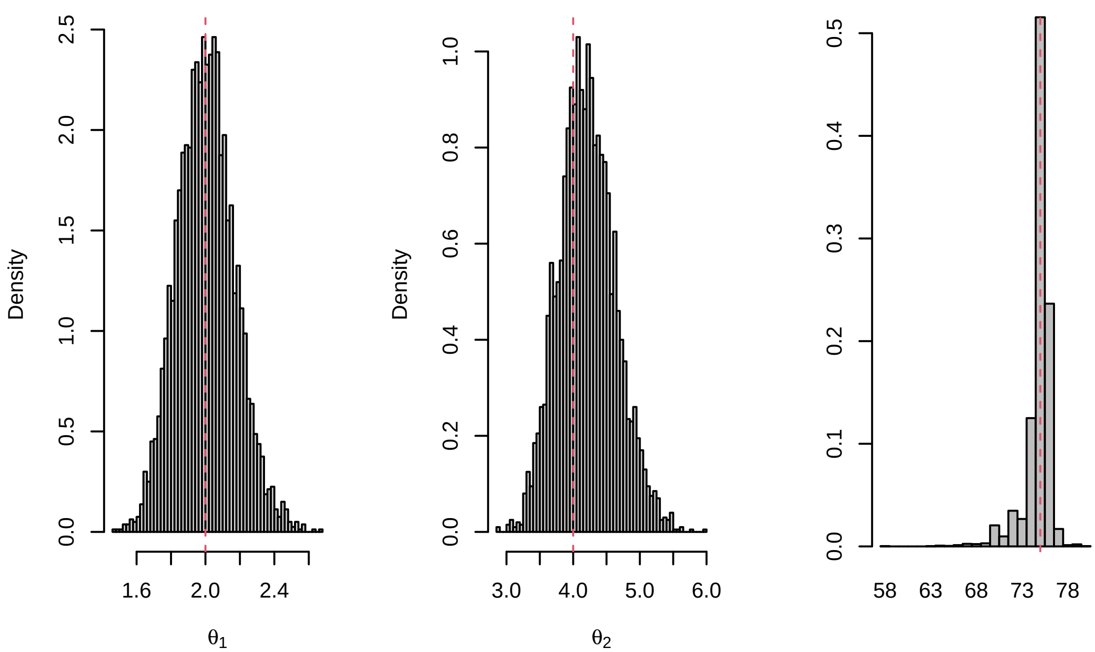
2: Simulate $\theta_2^{(k)} \sim \text{Gamma}\left(p_2 + S_n - S_{t^{(k-1)}}, q_2 + n - t^{(k-1)}\right)$.

3: Calculate the probability vector $\pi\left(t \mid \theta_1^{(k)}, \theta_2^{(k)}, x\right)$. Simulate a value $t^{(k)}$ from the set $\{1, 2, \ldots, n-1\}$ according to that probability vector.

```
    logprob = S[-n] * log(theta1[1]/theta2[1]) - (theta1[1] - theta2[1]) * (1:(n -
        1))
    t[1] = sample(n - 1, 1, prob = exp(logprob - max(logprob)))
    for (i in 2:niter) {
        theta1[i] = rgamma(1, p1 + S[t[i - 1]], q1 + t[i - 1])
        theta2[i] = rgamma(1, p2 + S[n] - S[t[i - 1]], q2 + n - t[i - 1])
        logprob = S[-n] * log(theta1[i]/theta2[i]) - (theta1[i] - theta2[i]) *
            (1:(n - 1))
        t[i] = sample(n - 1, 1, prob = exp(logprob - max(logprob)))
    }
    return(list(theta1 = theta1[-(1:nburn)], theta2 = theta2[-(1:nburn)], t = t[-(1:nburn)]))
}

n = 100
theta1 = 2
theta2 = 4
t = 75
Y = c(rpois(t, theta1), rpois(n - t, theta2))
plot(Y, type = "b", pch = 16, lwd = 2)
abline(h = theta1, col = 4, lty = 2)
abline(h = theta2, col = 4, lty = 2)
abline(v = t, col = 2, lty = 2)
```




```
posterior = MCMCchangepoint(Y, 1, 1, 0.5, 0, 0.5, 0, 5000, 1000)
par(mfrow = c(1, 3))
hist(posterior$theta1, "FD", freq = FALSE, main = NA, xlab = expression(theta[1]))
abline(v = theta1, col = 2, lty = 2)
hist(posterior$theta2, "FD", freq = FALSE, main = NA, xlab = expression(theta[2]))
abline(v = theta2, col = 2, lty = 2)
barplot(table(factor(posterior$t, levels = min(posterior$t):max(posterior$t)))/4000,
    space = 0)
abline(v = t - min(posterior$t) + 0.5, col = 2, lty = 2)
```



3.2 Mixture Model

Let $y_1, \dots, y_n$ be a random sample from the following mixture of Poisson distributions:

$$f(y_i \mid \alpha, \beta, \gamma) = \gamma f_{\text{Poisson}}(y_i \mid \alpha) + (1-\gamma) f_{\text{Poisson}}\left(y_i \mid \alpha e^{\beta x_i}\right),$$

where we denote the probability mass function of the distribution Poisson(θ) by $f_{\text{Poisson}}(y_i \mid \theta)$. We consider prior independence with prior distributions $\alpha \sim \text{Gamma}(2,1)$, $\beta \sim \mathcal{N}(0,1)$ and $\gamma \sim U(0,1) \equiv \text{Beta}(1,1)$.

a. Calculate the conditional posterior distributions of α, β, γ. Use the prior distributions of α, γ as independent proposal densities and a random walk proposal for the parameter β.

b. Now, consider the following data augmentation technique. For each y_i, we insert a binary random variable z_i such that:

$$P(Z_i = 1 \mid \gamma) = 1 - P(Z_i = 0 \mid \gamma) = \gamma.$$

Then, the conditional probability mass function of y_i given z_i is given by:

$$f(y_i \mid z_i, \alpha, \beta) = \begin{cases} f_{\text{Poisson}}(y_i \mid \alpha), & z_i = 1 \\ f_{\text{Poisson}}\left(y_i \mid \alpha e^{\beta x_i}\right), & z_i = 0 \end{cases}.$$

Calculate the conditional posterior distributions of all unknown quantities.

Solution.

a. The joint prior distribution of α, β and γ may be written as:

$$\begin{aligned} \pi(\alpha, \beta, \gamma) &= \pi(\alpha) \cdot \pi(\beta) \cdot \pi(\gamma) \\ &= \alpha e^{-\alpha} \cdot \frac{1}{\sqrt{2\pi}} e^{-\beta^2/2} \cdot 1 \\ &\propto \alpha e^{-\alpha} \cdot e^{-\beta^2/2}. \end{aligned}$$

The likelihood of the sample is given by:

$$\begin{aligned} f(y \mid \alpha, \beta, \gamma) &= \prod_{i=1}^n f(y_i \mid \alpha, \beta, \gamma) \\ &= \prod_{i=1}^n \left[\gamma f_{\text{Poisson}}(y_i \mid \alpha) + (1-\gamma) f_{\text{Poisson}}\left(y_i \mid \alpha e^{\beta x_i}\right)\right] \\ &= \prod_{i=1}^n \left[\gamma e^{-\alpha} \frac{\alpha^{y_i}}{y_i!} + (1-\gamma) e^{-\alpha e^{\beta x_i}} \frac{\alpha^{y_i} e^{\beta x_i y_i}}{y_i!}\right] \\ &= \prod_{i=1}^n \frac{\alpha^{y_i}}{y_i!} \left[\gamma e^{-\alpha} + (1-\gamma) e^{\beta x_i y_i - \alpha e^{\beta x_i}}\right] \\ &\propto \alpha^{n\bar{y}} \prod_{i=1}^n \left[\gamma e^{-\alpha} + (1-\gamma) e^{\beta x_i y_i - \alpha e^{\beta x_i}}\right]. \end{aligned}$$

For the calculation of the likelihood, we use the Log-Sum-Exp trick. In other words, we define:

$$v_{i1} = \log\gamma + \log f_{\text{Poisson}}(y_i \mid \alpha), \quad v_{i1} = \log(1-\gamma) + \log f_{\text{Poisson}}\left(y_i \mid \alpha e^{\beta x_i}\right), \quad m_i = \max\{v_{i1}, v_{i2}\}.$$

Then, we infer that:

$$\log f(y \mid \alpha, \beta, \gamma) = \sum_{i=1}^{n} \left[m_i + \log \left(e^{v_{i1} - m_i} + e^{v_{i2} - m_i} \right) \right].$$

Therefore, we get the joint posterior distribution of α, β and γ as follows:

$$\begin{aligned}
\pi(\alpha, \beta, \gamma \mid y) &\propto \pi(\alpha, \beta, \gamma) \cdot f(y \mid \alpha, \beta, \gamma) \\
&\propto \alpha e^{-\alpha} \cdot e^{-\beta^2/2} \cdot \alpha^{n\bar{y}} \prod_{i=1}^{n} \left[\gamma e^{-\alpha} + (1-\gamma) e^{\beta x_i y_i - \alpha e^{\beta x_i}} \right] \\
&= \alpha^{n\bar{y}+1} e^{-\alpha} \cdot e^{-\beta^2/2} \cdot \prod_{i=1}^{n} \left[\gamma e^{-\alpha} + (1-\gamma) e^{\beta x_i y_i - \alpha e^{\beta x_i}} \right].
\end{aligned}$$

We implement the following Metropolis-Hastings algorithm to simulate from this joint posterior distribution. We adjust the proposal variance σ_β^2 so that the percentage of accepted values for β is roughly equal to 50%.

Algorithm 3.2 Metropolis-Hastings

Initialize $\alpha^{(0)}$, $\gamma^{(0)}$, $\beta^{(0)}$.

Iterate the following steps:

1: Simulate $\alpha^* \sim \text{Gamma}(10, 1)$ and $U_\alpha \sim U(0,1)$.

2: Calculate the ratio:

$$A_\alpha = \frac{f\left(y \mid \alpha^*, \beta^{(\ell-1)}, \gamma^{(\ell-1)}\right)}{f\left(y \mid \alpha^{(\ell-1)}, \beta^{(\ell-1)}, \gamma^{(\ell-1)}\right)}.$$

3: If $U_\alpha < A_\alpha$, then let $\alpha^{(\ell)} = \alpha^*$. Otherwise, let $\alpha^{(\ell)} = \alpha^{(\ell-1)}$.

4: Simulate $\gamma^* \sim U(0,1)$ and $U_\gamma \sim U(0,1)$.

5: Calculate the ratio:

$$A_\gamma = \frac{f\left(y \mid \alpha^{(\ell)}, \beta^{(\ell-1)}, \gamma^*\right)}{f\left(y \mid \alpha^{(\ell)}, \beta^{(\ell-1)}, \gamma^{(\ell-1)}\right)}.$$

6: If $U_\gamma < A_\gamma$, then let $\gamma^{(\ell)} = \gamma^*$. Otherwise, let $\gamma^{(\ell)} = \gamma^{(\ell-1)}$.

7: Simulate $\beta^* \sim \mathcal{N}\left(\beta^{(\ell-1)}, \sigma_\beta^2\right)$ and $U_\beta \sim U(0,1)$.

8: Calculate the ratio:

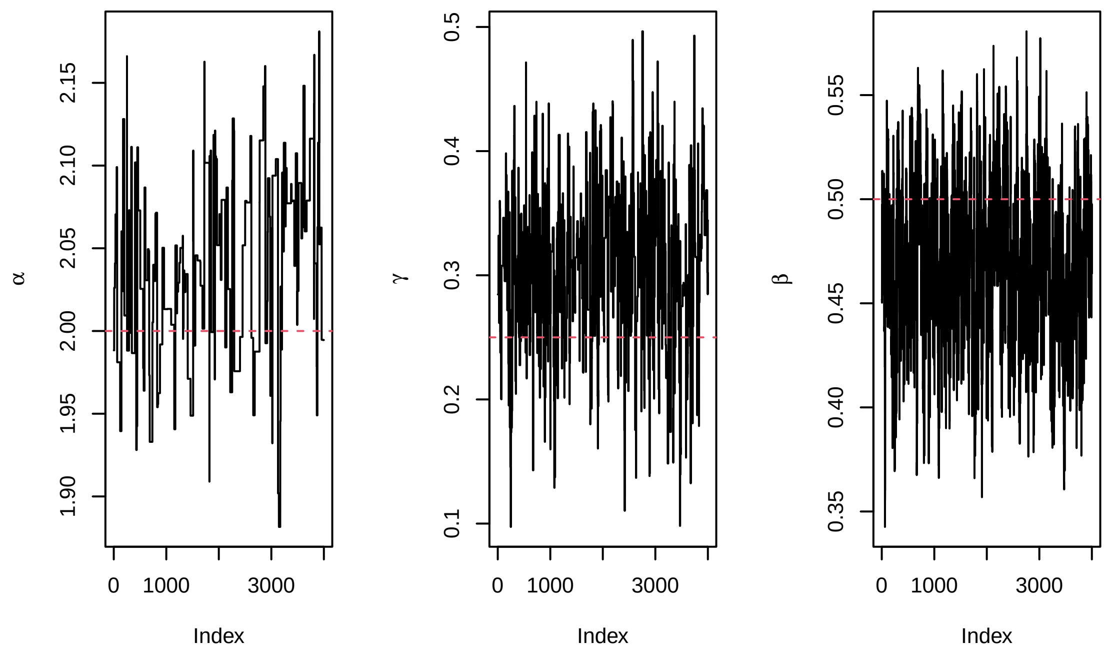
$$A_\beta = \frac{\pi\left(\beta^* \mid \alpha^{(\ell)}, \gamma^{(\ell)}, y\right)}{\pi\left(\beta^{(\ell-1)} \mid \alpha^{(\ell)}, \gamma^{(\ell)}, y\right)}.$$

9: If $U_\beta < A_\beta$, then let $\beta^{(\ell)} = \beta^*$. Otherwise, let $\beta^{(\ell)} = \beta^{(\ell-1)}$.

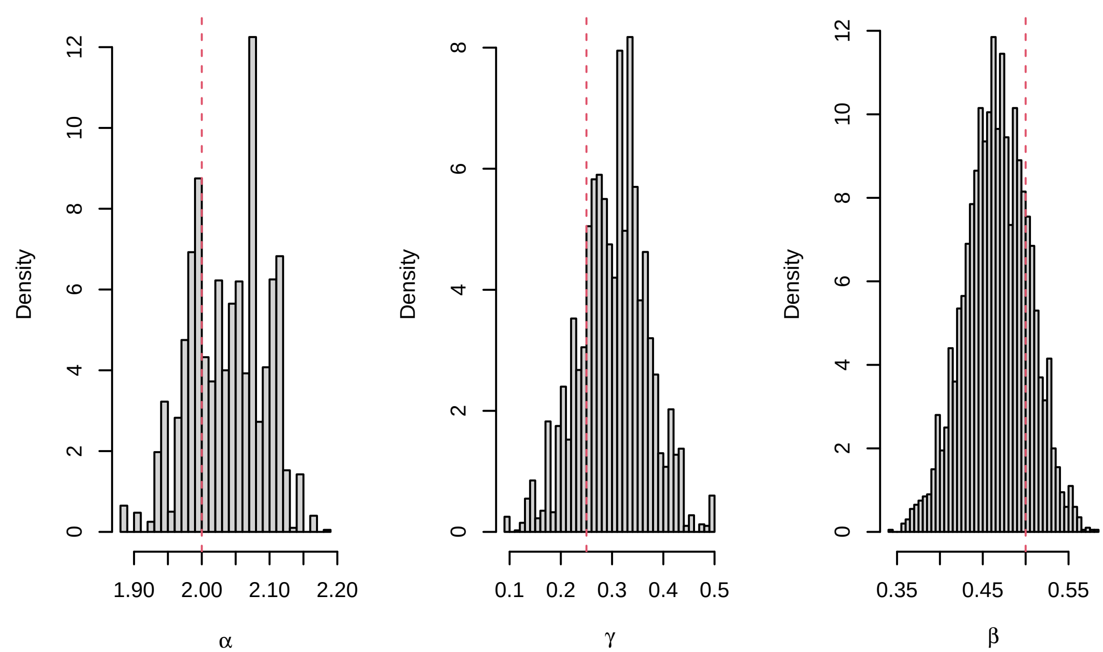
```
logdpois = function(Y, X, alpha, gamma, beta) {
    logprob = cbind(log(gamma) + dpois(Y, alpha, log = TRUE), log(1 - gamma) +
```

```
        dpois(Y, alpha * exp(beta * X), log = TRUE))
    maximum = apply(logprob, 1, max)
    return(sum(maximum + log(rowSums(exp(logprob - maximum)))))
}

MHpois = function(Y, X, alpha0, gamma0, beta0, betasd, niter, nburn) {
    alpha = numeric(niter)
    gamma = numeric(niter)
    beta = numeric(niter)
    alpha[1] = alpha0
    gamma[1] = gamma0
    beta[1] = beta0
    for (i in 2:niter) {
        alphastar = rgamma(1, 2)
        logA = logdpois(Y, X, alphastar, gamma[i - 1], beta[i - 1]) - logdpois(Y,
            X, alpha[i - 1], gamma[i - 1], beta[i - 1])
        alpha[i] = ifelse(log(runif(1)) < logA, alphastar, alpha[i - 1])
        gammastar = runif(1)
        logA = logdpois(Y, X, alpha[i], gammastar, beta[i - 1]) - logdpois(Y,
            X, alpha[i], gamma[i - 1], beta[i - 1])
        gamma[i] = ifelse(log(runif(1)) < logA, gammastar, gamma[i - 1])
        betastar = rnorm(1, beta[i - 1], betasd)
        logA = (beta[i - 1]^2 - betastar^2)/2 + logdpois(Y, X, alpha[i], gamma[i],
            betastar) - logdpois(Y, X, alpha[i], gamma[i], beta[i - 1])
        beta[i] = ifelse(log(runif(1)) < logA, betastar, beta[i - 1])
    }
    return(list(alpha = alpha[-(1:nburn)], gamma = gamma[-(1:nburn)], beta = beta[-(1:nburn)]))
}

n = 1000
alpha = 2
gamma = 0.25
beta = 0.5
X = rnorm(n)
Z = rbinom(n, 1, gamma)
Y = ifelse(Z == 1, rpois(n, alpha), rpois(n, alpha * exp(beta * X)))
posterior = MHpois(Y, X, 1, 0.5, 0, 0.05, 5000, 1000)
par(mfrow = c(1, 3))
plot(posterior$alpha, type = "l", ylab = expression(alpha))
abline(h = alpha, col = 2, lty = 2)
plot(posterior$gamma, type = "l", ylab = expression(gamma))
abline(h = gamma, col = 2, lty = 2)
plot(posterior$beta, type = "l", ylab = expression(beta))
```

```
abline(h = beta, col = 2, lty = 2)
```




```
hist(posterior$alpha, "FD", freq = FALSE, main = NA, xlab = expression(alpha))
abline(v = alpha, col = 2, lty = 2)
hist(posterior$gamma, "FD", freq = FALSE, main = NA, xlab = expression(gamma))
abline(v = gamma, col = 2, lty = 2)
hist(posterior$beta, "FD", freq = FALSE, main = NA, xlab = expression(beta))
abline(v = beta, col = 2, lty = 2)
```



We observe that the prior distribution of α isn't efficient as an independent proposal density.

b. We define:

$$n_1 = \sum_{i=1}^n \mathbb{1}_{\{z_i=1\}}, \quad S_{XY} = \sum_{i=1}^n \mathbb{1}_{\{z_i=0\}} x_i y_i, \quad S_\beta = \sum_{i=1}^n \mathbb{1}_{\{z_i=0\}} e^{\beta x_i}.$$

The complete-data likelihood, i.e. the joint likelihood of the observed variables y_i and the latent variables z_i, is given by:

$$\begin{aligned}
f(y, z \mid \alpha, \beta, \gamma) &= \prod_{i=1}^n f(y_i, z_i \mid \alpha, \beta, \gamma) \\
&= \prod_{i=1}^n f(z_i \mid \gamma) f(y_i \mid z_i, \alpha, \beta) \\
&= \prod_{i=1}^n \left[P(Z_i = 1 \mid \gamma) f_{\text{Poisson}}(y_i \mid \alpha)\right]^{\mathbb{1}_{\{z_i=1\}}} \left[P(Z_i = 0 \mid \gamma) f_{\text{Poisson}}\left(y_i \mid \alpha e^{\beta x_i}\right)\right]^{\mathbb{1}_{\{z_i=0\}}} \\
&= \prod_{i=1}^n \left(\gamma e^{-\alpha} \frac{\alpha^{y_i}}{y_i!}\right)^{\mathbb{1}_{\{z_i=1\}}} \left[(1-\gamma) e^{-\alpha e^{\beta x_i}} \frac{\alpha^{y_i} e^{\beta x_i y_i}}{y_i!}\right]^{\mathbb{1}_{\{z_i=0\}}} \\
&= \prod_{i=1}^n \left(\gamma e^{-\alpha}\right)^{\mathbb{1}_{\{z_i=1\}}} \left[(1-\gamma) e^{\beta x_i y_i - \alpha e^{\beta x_i}}\right]^{\mathbb{1}_{\{z_i=0\}}} \prod_{i=1}^n \frac{\alpha^{y_i}}{y_i!} \\
&\propto \gamma^{n_1} e^{-n_1 \alpha} \cdot (1-\gamma)^{n-n_1} e^{\beta S_{XY} - \alpha S_\beta} \cdot \alpha^{n\overline{y}} \\
&= \alpha^{n\overline{y}} e^{-(S_\beta + n_1)\alpha} \cdot \gamma^{n_1} (1-\gamma)^{n-n_1} \cdot e^{S_{XY}\beta}.
\end{aligned}$$

Therefore, we get the conditional posterior distributions of α and γ as follows:

$$\begin{aligned}
\pi(\alpha \mid \beta, \gamma, z, y) &\propto \pi(\alpha, \beta, \gamma, z \mid y) \\
&\propto \pi(\alpha, \beta, \gamma) \cdot f(y, z \mid \alpha, \beta, \gamma) \\
&\propto \alpha e^{-\alpha} \cdot \alpha^{n\overline{y}} e^{-(S_\beta + n_1)\alpha} \\
&= \alpha^{n\overline{y}+1} e^{-(S_\beta + n_1 + 1)\alpha},
\end{aligned}$$

$$\begin{aligned}
\pi(\gamma \mid \alpha, \beta, z, y) &\propto \pi(\alpha, \beta, \gamma) \cdot f(y, z \mid \alpha, \beta, \gamma) \\
&\propto \gamma^{n_1} (1-\gamma)^{n-n_1}.
\end{aligned}$$

In other words,

$$\alpha \mid \beta, z, y \sim \text{Gamma}\left(n\overline{y} + 2, S_\beta + n_1 + 1\right),$$

$$\gamma \mid z \sim \text{Beta}\left(n_1 + 1, n - n_1 + 1\right).$$

Furthermore, we get the conditional posterior distribution of β as follows:

$$\begin{aligned}
\pi(\beta \mid \alpha, z, y) &\propto \pi(\alpha, \beta, \gamma) \cdot f(y, z \mid \alpha, \beta, \gamma) \\
&\propto e^{-\beta^2/2 + S_{XY}\beta - \alpha S_\beta},
\end{aligned}$$

which isn't some known distribution.

Finally, we get the conditional posterior distribution of the latent variables z_i as follows:

$$f(z_i \mid y_i, \alpha, \beta, \gamma) \propto f(y_i, z_i \mid \alpha, \beta, \gamma) \propto \left(\gamma e^{-\alpha}\right)^{\mathbb{1}_{\{z_i=1\}}} \left[(1-\gamma) e^{\beta x_i y_i - \alpha e^{\beta x_i}}\right]^{\mathbb{1}_{\{z_i=0\}}}.$$

In other words,

$$(z_i \mid y_i, \alpha, \beta, \gamma) \sim \text{Bernoulli}\left(\frac{\gamma e^{-\alpha}}{\gamma e^{-\alpha} + (1-\gamma)e^{\beta x_i y_i - \alpha e^{\beta x_i}}}\right).$$

We implement the following Markov Chain Monte Carlo algorithm to simulate from this joint posterior distribution.

Algorithm 3.3 Markov Chain Monte Carlo

Initialize $\alpha^{(0)}$, $\gamma^{(0)}$, $\beta^{(0)}$, $z^{(0)}$.

Iterate the following steps:

1: Simulate $\alpha^{(\ell)} \sim \text{Gamma}\left(n\overline{y} + 2, S_{\beta^{(\ell-1)}} + n_1 + 1\right)$.

2: Simulate $\gamma^{(\ell)} \sim \text{Beta}\left(n_1 + 1, n - n_1 + 1\right)$.

3: Simulate $\beta^* \sim \mathcal{N}\left(\beta^{(\ell-1)}, \sigma_\beta^2\right)$ and $U_\beta \sim U(0,1)$.

4: Calculate the ratio:

$$A_\beta = \frac{\pi\left(\beta^* \mid \alpha^{(\ell)}, z^{(\ell-1)}, y\right)}{\pi\left(\beta^{(\ell-1)} \mid \alpha^{(\ell)}, z^{(\ell-1)}, y\right)}.$$

5: If $U_\beta < A_\beta$, then let $\beta^{(\ell)} = \beta^*$. Otherwise, let $\beta^{(\ell)} = \beta^{(\ell-1)}$.

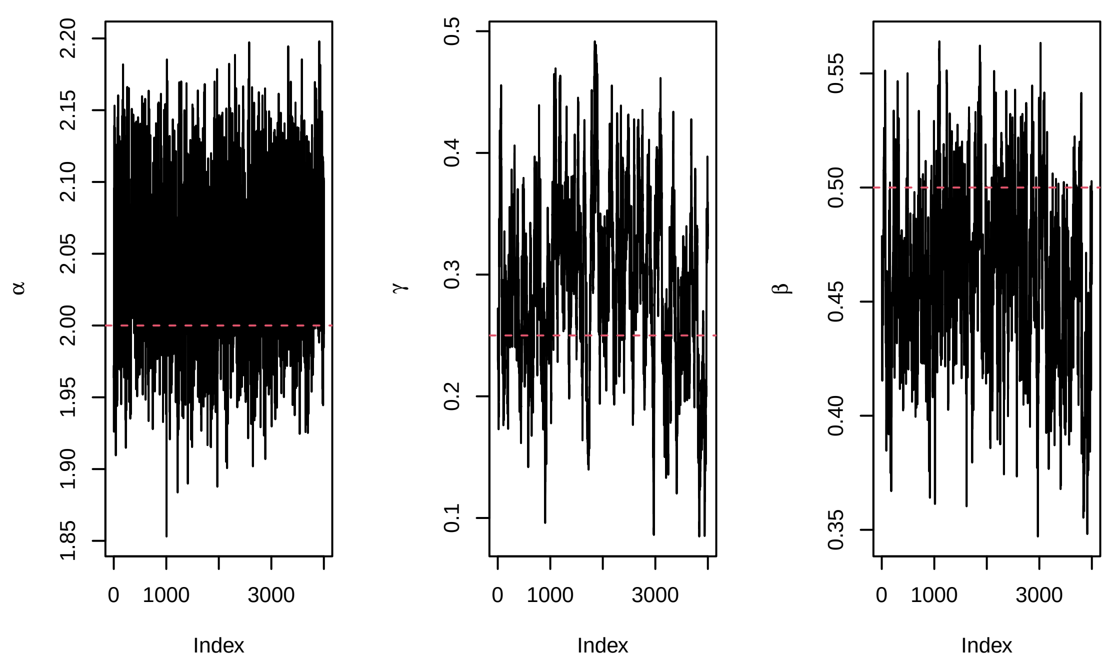
6: Calculate the probabilities $p_i = P\left(Z_i = 1 \mid y_i, \alpha^{(\ell)}, \gamma^{(\ell)}, \beta^{(\ell)}\right)$.

7: Simulate $U_i \sim U(0,1)$.

8: If $U_i < p_i$, then let $z_i^{(\ell)} = 1$. Otherwise, let $z_i^{(\ell)} = 0$.

```
prob = function(Y, X, alpha, gamma, beta) {
    logprob = cbind(log(gamma) - alpha, log(1 - gamma) + beta * X * Y - alpha *
        exp(beta * X))
    maximum = apply(logprob, 1, max)
    unnormalized = exp(logprob - maximum)
    return(unnormalized[, 1]/rowSums(unnormalized))
}

MCMCpois = function(Y, X, alpha0, gamma0, beta0, betasd, niter, nburn) {
    n = length(Y)
    S = sum(Y)
    alpha = numeric(niter)
    gamma = numeric(niter)
    beta = numeric(niter)
    Z = matrix(0, niter, n)
    alpha[1] = alpha0
    gamma[1] = gamma0
    beta[1] = beta0
    Z[1, ] = rbinom(n, 1, prob(Y, X, alpha[1], gamma[1], beta[1]))
    for (i in 2:niter) {
        n1 = sum(Z[i - 1, ])
```

```
        Sbeta = sum(exp(beta[i - 1] * X[Z[i - 1, ] == 0]))
        alpha[i] = rgamma(1, S + 2, Sbeta + n1 + 1)
        gamma[i] = rbeta(1, n1 + 1, n - n1 + 1)
        betastar = rnorm(1, beta[i - 1], betasd)
        logA = (beta[i - 1]^2 - betastar^2)/2 + sum(X[Z[i - 1, ] == 0] * Y[Z[i -
            1, ] == 0]) * (betastar - beta[i - 1]) + alpha[i] * (Sbeta - sum(exp(betastar *
            X[Z[i - 1, ] == 0])))
        beta[i] = ifelse(log(runif(1)) < logA, betastar, beta[i - 1])
        Z[i, ] = rbinom(n, 1, prob(Y, X, alpha[i], gamma[i], beta[i]))
    }
    return(list(alpha = alpha[-(1:nburn)], gamma = gamma[-(1:nburn)], beta = beta[-(1:nburn)],
        Z = Z[-(1:nburn), ]))
}

posterior = MCMCpois(Y, X, 1, 0.5, 0, 0.05, 5000, 1000)
par(mfrow = c(1, 3))
plot(posterior$alpha, type = "l", ylab = expression(alpha))
abline(h = alpha, col = 2, lty = 2)
plot(posterior$gamma, type = "l", ylab = expression(gamma))
abline(h = gamma, col = 2, lty = 2)
plot(posterior$beta, type = "l", ylab = expression(beta))
abline(h = beta, col = 2, lty = 2)
```




```
hist(posterior$alpha, "FD", freq = FALSE, main = NA, xlab = expression(alpha))
abline(v = alpha, col = 2, lty = 2)
hist(posterior$gamma, "FD", freq = FALSE, main = NA, xlab = expression(gamma))
abline(v = gamma, col = 2, lty = 2)
```

```
hist(posterior$beta, "FD", freq = FALSE, main = NA, xlab = expression(beta))
abline(v = beta, col = 2, lty = 2)
```